



Chapter 4

Asymmetric Cryptography

- Introduction
- Encryption: RSA
- Key Exchange: Diffie-Hellman

Design of Asymmetric Cryptosystems



- Difficulty: Find an **encryption algorithm** and a **key generating method** to construct two keys $-K$, $+K$ such that it is not possible to decipher $E(+K, m)$ with the knowledge of $+K$
 - Constraints:
 - The key length should be “manageable”
 - Encrypted messages should not be arbitrarily longer than unencrypted messages (we would tolerate a small constant factor)
 - Encryption and decryption should not consume too much resources (time, memory)
 - Basic idea: Take a problem in the area of mathematics / computer science, that is *hard* to solve when knowing only $+K$, but *easy* to solve when knowing $-K$
 - Knapsack problems: basis of first working algorithms, which were unfortunately almost all proven to be insecure
 - Factorization problem: basis of the **RSA algorithm**
 - Discrete logarithm problem: basis of **Diffie-Hellman** and ElGamal

- General idea:
 - Use two different keys $-K$ and $+K$ for encryption and decryption
 - Given a random ciphertext $c = E(+K, m)$ and $+K$ it should be infeasible to compute $m = D(-K, c) = D(-K, E(+K, m))$
 - This implies that it should be infeasible to compute $-K$ when given $+K$
 - The key $-K$ is only known to one entity A and is called A's **private key $-K_A$**
 - The key $+K$ can be publicly announced and is called A's **public key $+K_A$**
- Applications:
 - **Encryption:** If B encrypts a message with A's public key $+K_A$, he can be sure that only A can decrypt it using $-K_A$
 - **Signing:** If A encrypts a message with his own private key $-K_A$, everyone can verify this signature by decrypting it with A's public key $+K_A$
- Attention: It is crucial, that everyone can verify that he really knows A's public key and not the key of an adversary!

RSA – Mathematical Background (Modular Arithmetic)



- We say b is congruent a mod n if it has the same remainder like a when divided by n . So, n divides $(a-b)$, and we write $b \equiv a \pmod{n}$
 - E.g., $4 \equiv 11 \pmod{7}$, $25 \equiv 11 \pmod{7}$
- Greatest common divisor
 - Let $a, b \in \mathbb{Z}$ and $d = \gcd(a, b)$. Then there exists $m, n \in \mathbb{Z}$ such that: $d = m \times a + n \times b$
- Euler totient of n : $\Phi(n)$
 - Let $\Phi(n)$ denote the number of positive integers less than n and relatively prime to n
 - Examples: $\Phi(4) = 2$, $\Phi(15) = 8$
 - If p is prime $\Rightarrow \Phi(p) = p - 1$
 - Let n and b be positive and relatively prime integers, i.e. $\gcd(n, b) = 1 \Rightarrow b^{\Phi(n)} \equiv 1 \pmod{n}$
- Euclidean Algorithm
 - The algorithm *Euclid* given a, b computes $\gcd(a, b)$
 - ```
int Euclid(int a, b) {
 if (b = 0) { return(a); }
 return(Euclid(b, a MOD b);
 }
```
- Extended Euclidean Algorithm
  - The algorithm *ExtEuclid* given  $a, b$  computes  $d, m, n$  such that:  $d = \gcd(a, b) = m \times a + n \times b$
  - ```
struct(int d, m, n) ExtEuclid(int a, b) {
    int d', m', n';
    if (b = 0) {return(a, 1, 0); }
    (d', m', n') = ExtEuclid(b, a MOD b);
    (d, m, n) = (d', m', m' - [a / b] * n');
    return(d, m, n);
  }
```

For more information, please refer to undergraduate CS classes or to the NetSec slides WS 2006/2007



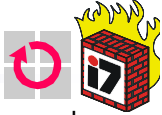
- ❑ Invented by R. Rivest, A. Shamir and L. Adleman [RSA78]
- ❑ **Key generation**
 - ❑ Select p, q p and q both prime, $p \neq q$
 - ❑ Calculate n $n = p \times q$
 - ❑ Calculate $\Phi(n)$ $\Phi(n) = (p - 1)(q - 1)$
 - ❑ Select integer e $\text{gcd}(\Phi(n), e) = 1; 1 < e < \Phi(n)$
 - ❑ Calculate d $d \times e \text{ mod } \Phi(n) = 1$ (extended Euclid)
 - ❑ Public key $+K = \{e, n\}$
 - ❑ Private key $-K = \{d, n\}$

- ❑ **Encryption**
 - ❑ Plaintext $M < n$ (what about 0, 1, ...?)
 - ❑ Ciphertext $C = M^e \text{ mod } n$

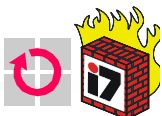
- ❑ **Decryption**
 - ❑ Ciphertext C
 - ❑ Plaintext $M = C^d \text{ mod } n$



- ❑ As $(d \times e) = (e \times d)$ the operation also works in the opposite direction, that means you can encrypt with d and decrypt with e
- ❑ This property allows to use the same keys d and e for:
 - ❑ Receiving messages that have been encrypted with one's public key
 - ❑ Sending messages that have been signed with one's private key



- ❑ Let p, q be distinct large primes and $n = p \times q$. Assume, we have also two integers e and d such that $d \times e \equiv 1 \text{ mod } \Phi(n)$
- ❑ Let M be an integer that represents the message to be encrypted, with M positive, smaller than and relatively prime to n .
 - ❑ Example: Encode with <blank> = 99, A = 10, B = 11, ..., Z = 35
So "HELLO" would be encoded as 1714212124.
If necessary, break M into blocks of smaller messages: 17142 12124
- ❑ To encrypt, compute: $C = M^e \text{ MOD } n$
 - ❑ This can be done efficiently using the *square-and-multiply algorithm*
- ❑ To decrypt, compute: $M' = C^d \text{ MOD } n$
- ❑ Proof
 $d \times e \equiv 1 \text{ mod } \Phi(n) \Rightarrow \exists k \in \mathbb{Z}: (d \times e) - 1 = k \times \Phi(n) \Leftrightarrow (d \times e) = k \times \Phi(n) + 1$
we have: $M' \equiv E^d \equiv M^{(e \times d)} \equiv M^{(k \times \Phi(n) + 1)} \equiv 1^k \times M \equiv M \text{ mod } n$ ■



- ❑ The security of the scheme lies in the difficulty of factoring $n = p \times q$ as it is easy to compute $\Phi(n)$ and then d , when p and q are known
- ❑ This class will not teach why it is difficult to factor large n 's, as this would require to dive deep into mathematics
 - ❑ If p and q fulfill certain properties, the best known algorithms are exponential in the number of digits of n
 - Please be aware that if you choose p and q in an "unfortunate" way, there might be algorithms that can factor more efficiently and your RSA encryption is not at all secure:
 - Thus, p and q should be about the same bit length and sufficiently large
 - $(p - q)$ should not be too small
 - If you want to choose a small encryption exponent, e.g. 3, there might be additional constraints, e.g. $\text{gcd}(p - 1, 3) = 1$ and $\text{gcd}(q - 1, 3) = 1$
 - The security of RSA also depends on the primes generated being truly random (like every key creation method for any algorithm)
 - ❑ Moral: If you are to implement RSA by yourself, ask a mathematician or better a cryptographer to check your design



Side channel attacks

- Optimizations for use of RSA in embedded systems depend on the Chinese remainder theorem (CRT)
 - Applications
 - Smart cards (token, banking)
 - Pay-per-view TV
 - and many others...
 - Use (and storage) of p and q allows to calculate $m^e \bmod p$, which can be efficiently manipulated to compute $m^e \bmod n$
 - Introducing computation errors allows to reveal the prime p
 $p = \gcd(s'-s, n)$ with s' and s being the bogus and correct signatures
- Implementation using square and multiply
 - Most RSA implementations rely on the square-and-multiply algorithm for the exponentiations
 - Timing attacks can be used to “guess” the private key

[A. G. Voyatzis, “An Introduction to Side Channel Cryptanalysis of RSA”, ACM Crossroads, vol. 11.3, 2004]

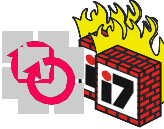
Diffie-Hellman – Mathematical Background



Definition: **discrete logarithm**

- Let p be prime, g be a primitive root of $(\mathbb{Z}_p^*, \times_p)$ and c be any element of \mathbb{Z}_p^* . Then there exists z such that: $g^z \equiv c \pmod p$
 z is called the *discrete logarithm* of c modulo p to the base g
- Example:
 - 6 is the discrete logarithm of 1 modulo 7 to the base 3 as $3^6 \equiv 1 \pmod 7$
- The calculation of the discrete logarithm z when given g , c , and p is a computationally difficult problem and the asymptotical runtime of the best known algorithms for this problem is exponential in the bit length of p

For more information, please refer to undergraduate CS classes or to the NetSec slides WS 2006/2007



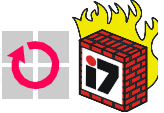
Finite groups

- Abelian group (S, \oplus) : set S and a binary operation \oplus with the following properties: closure, identity, associativity, commutativity and inverse elements
- Finite group: Abelian group plus finite set of elements, i.e. $|S| < \infty$

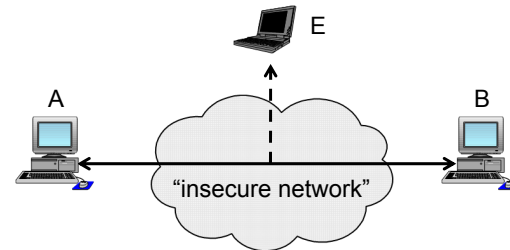
Primitive root, generator

- Let (S, \bullet) be a group, $g \in S$ and $g^a := g \bullet g \bullet \dots \bullet g$ (a times with $a \in \mathbb{Z}^+$)
 Then g is called a *primitive root* of $(S, \bullet) \Leftrightarrow \{g^a \mid 1 \leq a \leq |S|\} = S$
- Examples:
 - 1 is a primitive root of $(\mathbb{Z}_n, +_n)$
 - 3 is a primitive root of $(\mathbb{Z}_7^*, \times_7)$
- $(\mathbb{Z}_n^*, \times_n)$ does have a primitive root $\Leftrightarrow n \in \{2, 4, p, 2 \times p^e\}$ where p is an odd prime and $e \in \mathbb{Z}^+$

Diffie-Hellman Key Exchange



- The Diffie-Hellman key exchange was first published in the landmark paper [DH76], which also introduced the fundamental idea of asymmetric cryptography
- The DH exchange in its basic form enables two parties A and B to agree upon a *shared secret* using a public channel:
 - *Public channel* means, that a potential attacker E (E stands for eavesdropper) can **read** all messages exchanged between A and B



Key Exchange Procedure



- A chooses a prime p , a primitive root g of Z_p^* , and a random number q :
 - A and B can agree upon the values p and g prior to any communication, or A can choose p and g and send them with his first message
 - A computes $v = g^q \text{ MOD } p$ and sends to B: $\{p, g, v\}$
- B chooses a random number r :
 - B computes $w = g^r \text{ MOD } p$ and sends to A: $\{p, g, w\}$ (or just $\{w\}$)
- Both sides compute the common secret:
 - A computes $s = w^q \text{ MOD } p$
 - B computes $s' = v^r \text{ MOD } p$
- As $g^{(q \times r)} \text{ MOD } p = g^{(r \times q)} \text{ MOD } p$ it holds: $s = s'$
- Remark: In practice the number g does not necessarily need to be a primitive root of p , it is sufficient if it generates a large subgroup of Z_p^*

Diffie-Hellman Key Exchange – Man-in-the-middle attack



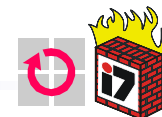
- Eve generates to random numbers q' and r' :
 - Eve computes $v' = g^{q'} \text{ MOD } p$ and $w' = g^{r'} \text{ MOD } p$
- When A sends $\{p, g, v\}$ she intercepts the message
 - Then, E sends to B: $\{p, g, v'\}$
- When B sends $\{p, g, w\}$ she intercepts the message as well
 - E sends to A: $\{p, g, w'\}$
- When the supposed “shared secret” is computed we get:
 - A computes $s_1 = w'^q \text{ MOD } p = v'^r \text{ MOD } p$ the latter computed by E
 - B computes $s_2 = v'^r \text{ MOD } p = w'^q \text{ MOD } p$ the latter computed by E
 - So, in fact A and E have agreed upon a shared secret s_1 , similarly E and B have agreed upon a shared secret s_2
- E can now use the “shared secret” to intercept all the messages encrypted by this key to forge and re-encrypt the messages without being noticed

Diffie-Hellman Key Exchange



- The mathematical basis for the DH exchange is the problem of finding *discrete logarithms in finite fields*
 - An attacker Eve (E) who is listening to the public channel can only compute the secret s , if she is able to compute either q or r which are the discrete logarithms of v, w modulo p to the base g
- It is important, that A and B can be sure, that the attacker is not able to **alter** messages, as in this case he might launch a *man-in-the-middle attack*
- Remark: The DH exchange is *not* an asymmetric encryption algorithm, but is nevertheless introduced here as it goes well with the mathematical flavor of this lecture... :o)

Diffie-Hellman Key Exchange



- Two countermeasures against the man-in-the-middle attack:
 - The shared secret is “*authenticated*” after it has been agreed upon
 - We will treat this in the section on key management
 - A and B use a so-called *interlock protocol* after agreeing on a shared secret:
 - For this they have to exchange messages that E has to relay before she can decrypt / re-encrypt them
 - The content of these messages has to be checkable by A and B
 - This forces E to invent messages and she can be detected
 - One technique to prevent E from decrypting the messages is to split them into two parts and to send the second part before the first one.
 - If the encryption algorithm used inhibits certain characteristics E can not encrypt the second part before she receives the first one.
 - As A will only send the first part after he received an answer (the second part of it) from B, E is forced to invent two messages, before she can get the first parts.



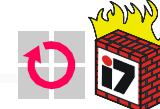
- ❑ Asymmetric cryptography allows to use two different keys for:
 - ❑ Encryption / Decryption
 - ❑ Signing / Verifying
- ❑ The most practical algorithms that are still considered to be secure are:
 - ❑ RSA, based on the difficulty of factoring
 - ❑ Diffie-Hellman (not an asymmetric algorithm, but a key agreement protocol)
 - ❑ ElGamal, like DH based on the difficulty of computing discrete logarithms
- ❑ As their security is entirely based on the difficulty of certain mathematical problems, algorithmic advances constitute their biggest threat

- ❑ Practical considerations:
 - ❑ Asymmetric cryptographic operations are magnitudes slower than symmetric ones
 - ❑ Therefore, they are often not used for encrypting / signing bulk data
 - ❑ Symmetric techniques are used to encrypt / compute a cryptographic hash value and asymmetric cryptography is just used to encrypt a key / hash value

Additional References



- [Bre88a] D. M. Bressoud. *Factorization and Primality Testing*. Springer, 1988.
- [Cor90a] T. H. Cormen, C. E. Leiserson, R. L. Rivest. *Introduction to Algorithms*. The MIT Press, 1990.
- [DH76] W. Diffie, M. E. Hellman. *New Directions in Cryptography*. IEEE Transactions on Information Theory, IT-22, pp. 644-654, 1976.
- [ElG85a] T. ElGamal. *A Public Key Cryptosystem and a Signature Scheme based on Discrete Logarithms*. IEEE Transactions on Information Theory, Vol.31, Nr.4, pp. 469-472, July 1985.
- [Kob87a] N. Koblitz. *A Course in Number Theory and Cryptography*. Springer, 1987.
- [Men93a] A. J. Menezes. *Elliptic Curve Public Key Cryptosystems*. Kluwer Academic Publishers, 1993.
- [Niv80a] I. Niven, H. Zuckerman. *An Introduction to the Theory of Numbers*. John Wiley & Sons, 4th edition, 1980.
- [RSA78] R. Rivest, A. Shamir und L. Adleman. *A Method for Obtaining Digital Signatures and Public Key Cryptosystems*. Communications of the ACM, February 1978.



- ❑ Principles of asymmetric cryptography
 - ❑ +K, -K for encryption and signing
 - ❑ Mathematical problems that are hard to solve
 - ❑ Factorization, discrete logarithm

- ❑ RSA
 - ❑ Key generation
 - ❑ Encryption / decryption (how?, why does it work?)

- ❑ Diffie-Hellman key exchange
 - ❑ Key generation procedure
 - ❑ Man-in-the-middle attack