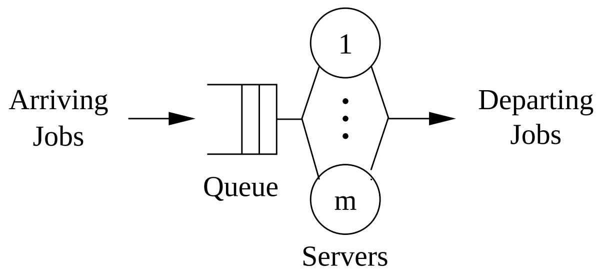
D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A =$ mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B =$ mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time

  B distribution of the service time

  m number of servers

◆ Special distributions of ***A*** and ***B*** :

- ➤ *M* exponential distribution
- ➤ E_k Erlang-*k*-distribution
- ➤ H_k hyperexponential distribution
- ➤ *D* deterministic distribution (T_A , T_B constant)
- ➤ *G* general distribution
- ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
- ➤ *SIRO* (Service-In-Random-Order)
- ➤ *RR* (Round Robin)
- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers $m = 1$.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again
(**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

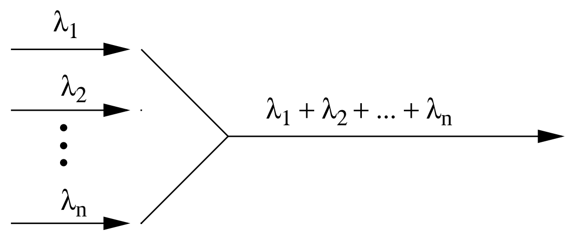
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \dfrac{1}{\lambda}\,,$
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

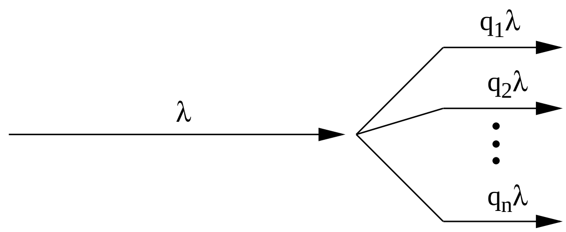
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

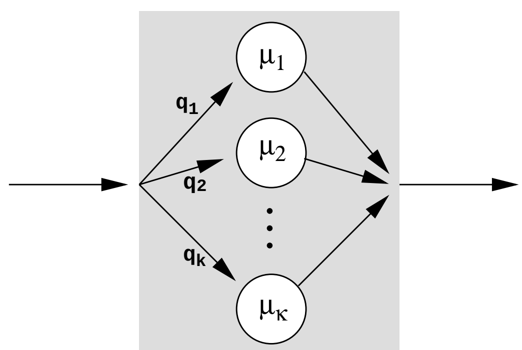


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

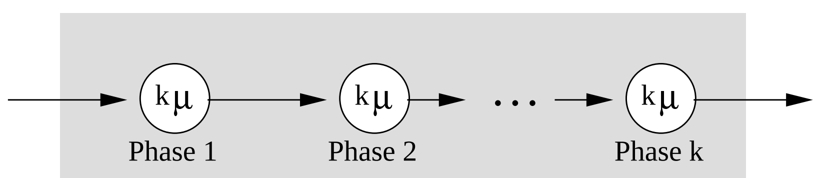
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\text{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots.$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots\ ,$$
$$\overline{X} = \frac{1}{\mu}\,,$$
$$\mathrm{var}(X) = \frac{1}{k\mu^2}\,,$$
$$c_X = \frac{1}{\sqrt{k}} \leq 1\,.$$

■ Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\,,$$
$$\mathrm{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\,,$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

- Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

- Parameters:

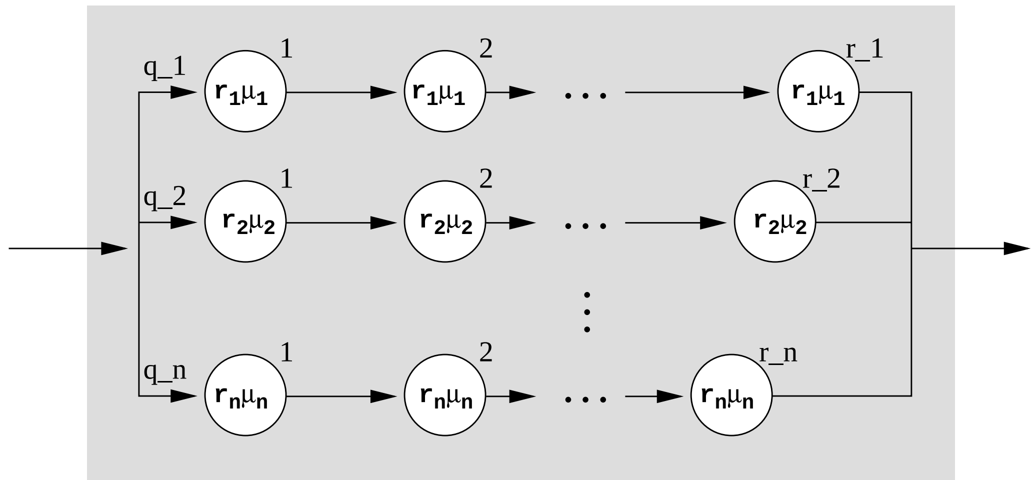
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

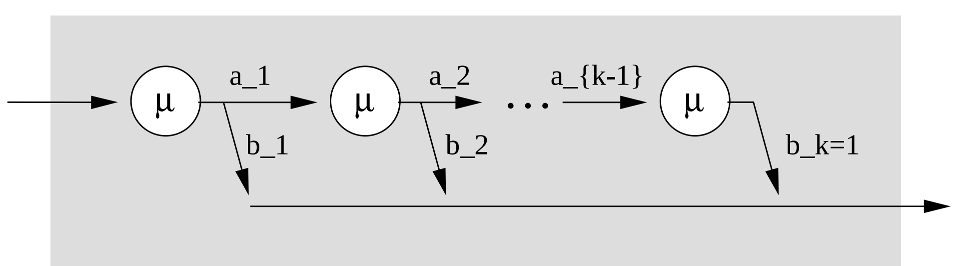
Generalized Erlang Distribution:



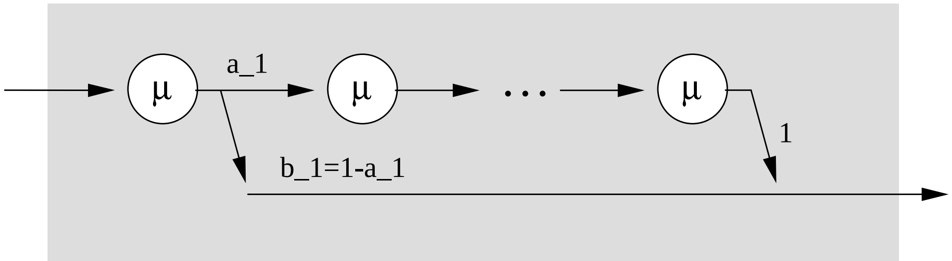
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j \mu_j (r_j \mu_j x)^{r_j - 1}}{(r_j - 1)!} \cdot e^{-r_j \mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



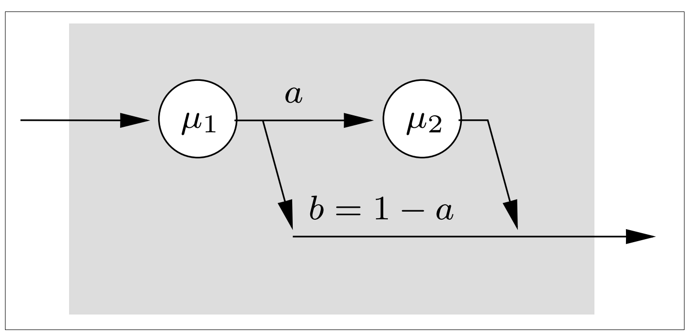
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu}\,,$$

$$\text{var}\,(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2}\,,$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1-b_1)]^2}\,.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2}\,,$$

$$\text{var}\,(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2}\,,$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}\,.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2+4-4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

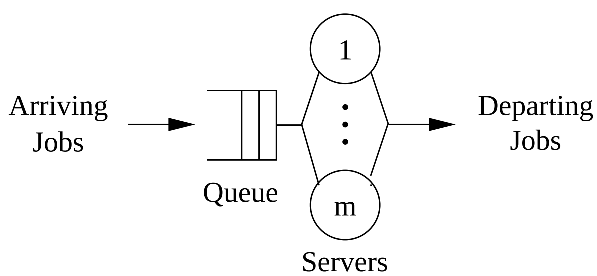
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2 ,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}] .$$

Utilization ρ:

- Eine Bedieneinheit (single server) *m = 1*:

$$\rho = \frac{\lambda}{\mu}.$$

- Multiple server *m > 1*:

  mμ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

- Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

Throughput λ:

- Number of served jobs per time unit (departure rate)
- If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

Response time *T*:

- Total time in system (queue + server)
- Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little’s theorem (Little’s law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \,,$$
$$\overline{Q} = \lambda \overline{W} \,.$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu},$$

Little's Law:

$$\overline{K} = \lambda \overline{T},$$

$$\overline{Q} = \lambda \overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

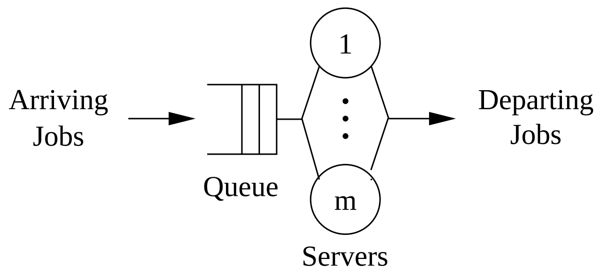
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
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  - ➤ number of Servers: *m*

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◆ Parameters:

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mean	$\overline{X} = \frac{1}{\lambda}$,
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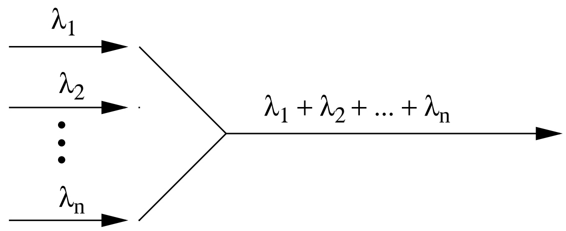
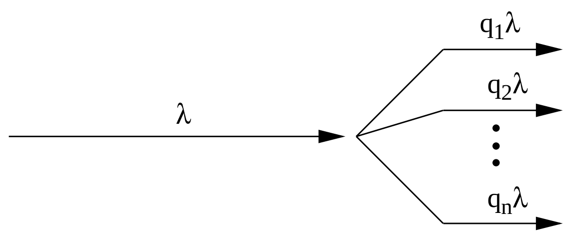
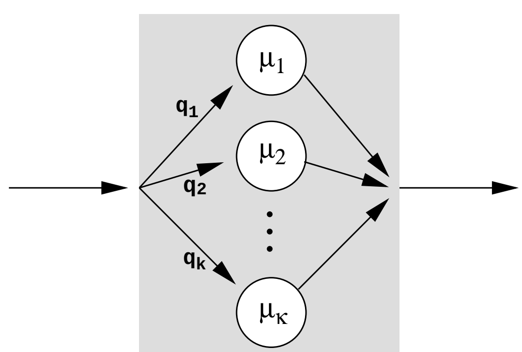


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



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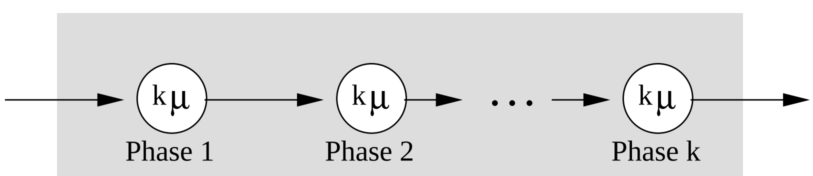
Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots .$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots ,$$
$$\overline{X} = \frac{1}{\mu},$$
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■ Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \ge 0.$$

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$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0 ,$$
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Gamma Distribution:

- Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

- Parameters:

$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

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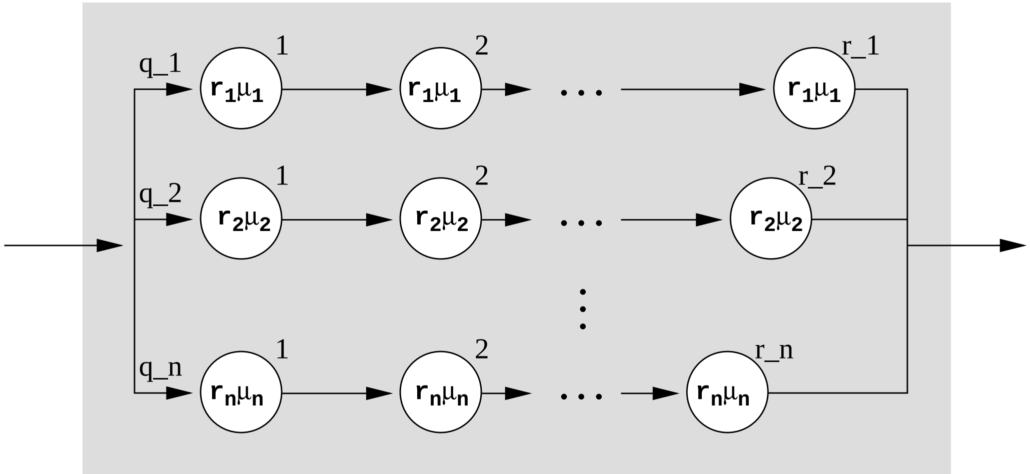
$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

Generalized Erlang Distribution:



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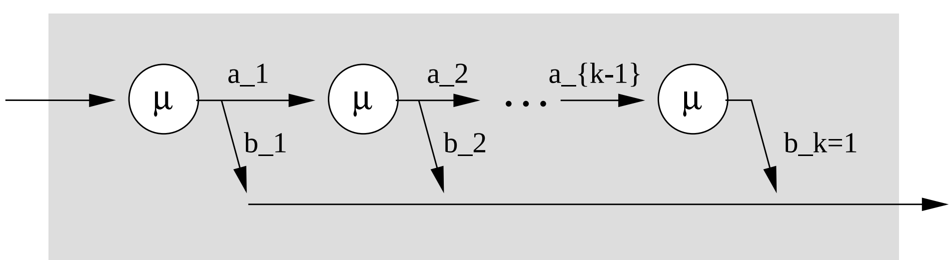
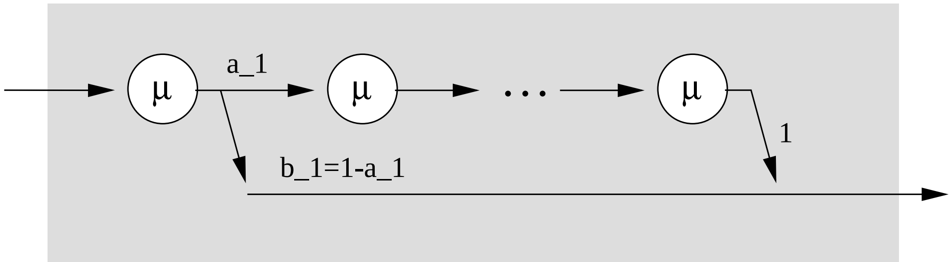
Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



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$$\operatorname{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k)+k-2)}{\mu^2},$$

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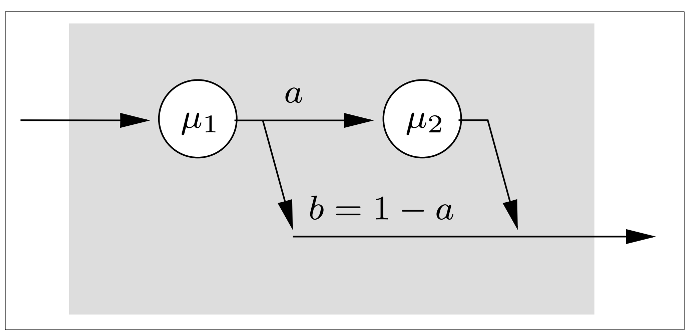
$C_X > 1$:



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$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

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$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha), \quad x \geq 0.$$

- Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha), \ \ \lambda > 0,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1+\frac{1}{\alpha}\right),$$

$$c_X^2 = \frac{\Gamma(1+2/\alpha)}{\{\Gamma(1+1/\alpha)\}^2} - 1.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $k=1,2,\ldots$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $(0<\alpha<\infty)$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2+\mu_2^2}}{\mu_1+\mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k}\frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k}\frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2\sum_{i=1}^{k}\frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	$\underset{k=1,2,\ldots}{\mu, k}$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	$\underset{0<\alpha<\infty}{\mu, \alpha}$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

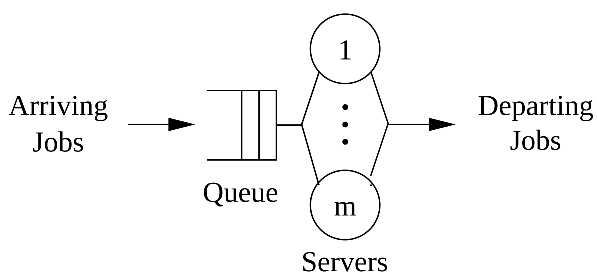
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:
  - Eine Bedieneinheit (single server) $m = 1$:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server $m > 1$:

    $m\mu$ = service rate of m servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:
  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time T:
  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:

  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:

  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T}\,,$$
$$\overline{Q} = \lambda \overline{W}\,.$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu},$$

Little's Law:

$$\overline{K} = \lambda\overline{T},$$
$$\overline{Q} = \lambda\overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

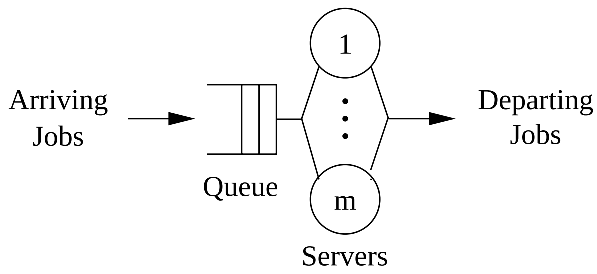
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A =$ mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B$ = mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- *FCFS* (First-Come-First-Served)
- *LCFS* (Last-Come-First-Served)
- *SIRO* (Service-In-Random-Order)
- *RR* (Round Robin)
- *PS* (Processor Sharing)
- IS (Infinite Server)
- Time independent priorities
- Time dependent priorities
- Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- Interarrival time ist **exponentially** distributed
- Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- Number of servers ***m* = *1***.
- Queueing dicipline: **LCFS**
- **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

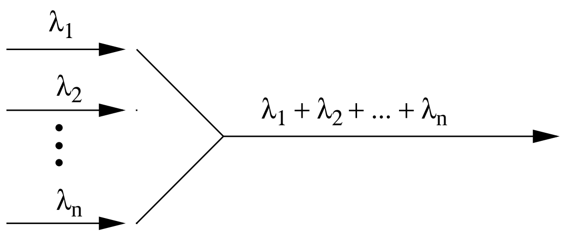
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}$,
mean	$\overline{X} = \dfrac{1}{\lambda}$,
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}$,
coefficient of variation	$c_X = 1$.

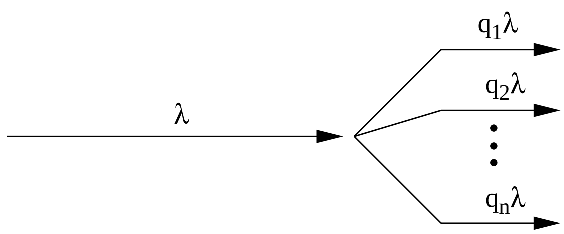
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

- Merging:

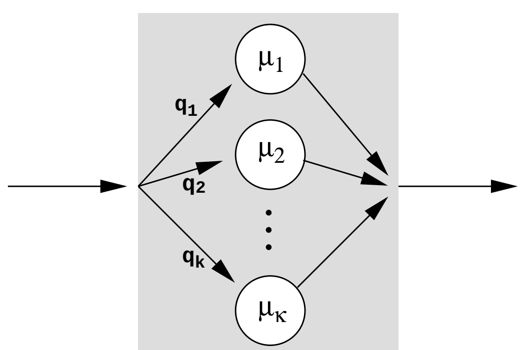


- Splitting:



Hyperexponential Distribution, H_k

- Model:



- Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

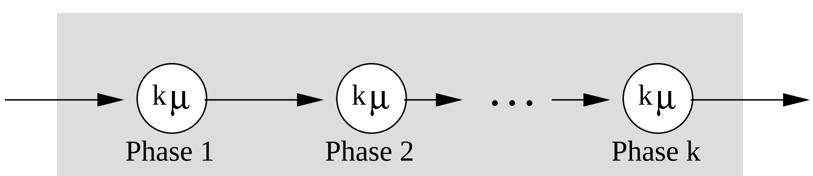
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\text{var}(X) = 2\sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \quad \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots.$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots\ ,$$
$$\overline{X} = \frac{1}{\mu}\,,$$
$$\mathrm{var}(X) = \frac{1}{k\mu^2}\,,$$
$$c_X = \frac{1}{\sqrt{k}} \le 1\,.$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \ge 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\,,$$
$$\mathrm{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\,,$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

◆ Parameters:

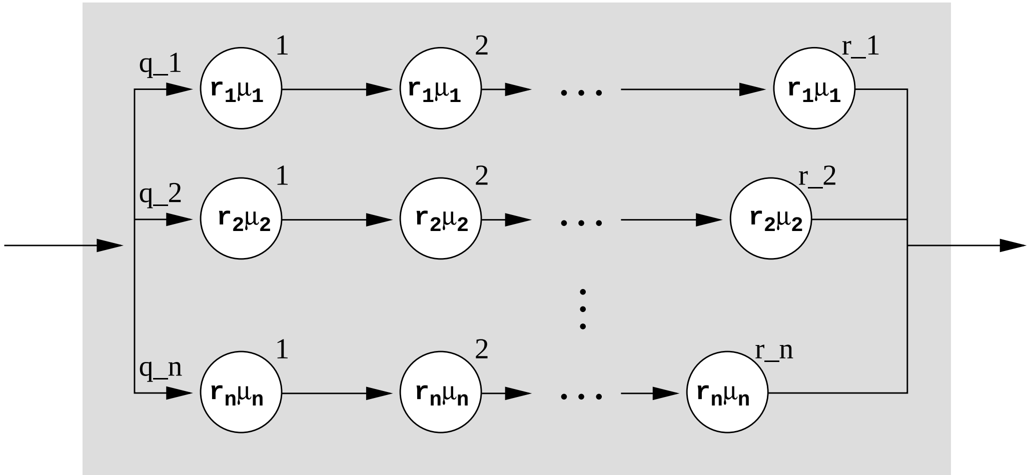
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

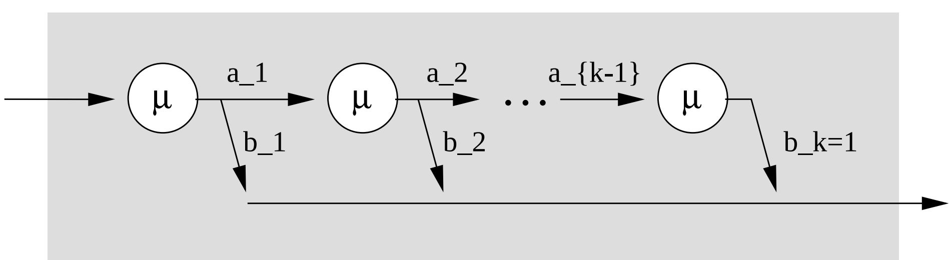
Generalized Erlang Distribution:



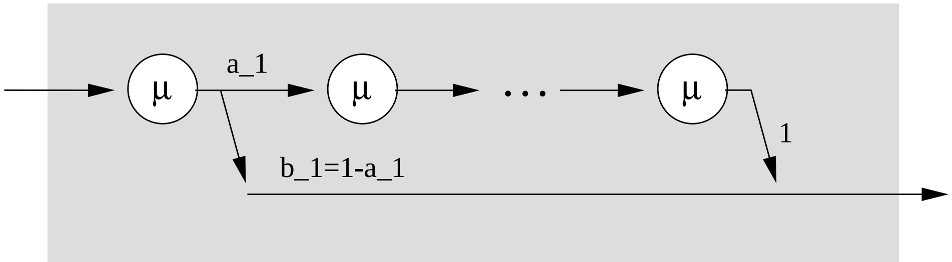
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



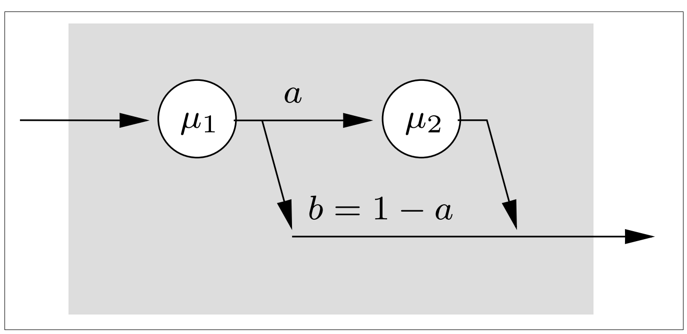
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\mathrm{var}(X) = \frac{k + b_1(k-1)\left(b_1(1-k) + k - 2\right)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\left(b_1(1-k) + k - 2\right)}{[b_1 + k(1-b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\mathrm{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1+2/\alpha)}{\{\Gamma(1+1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\text{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \mathrm{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \mathrm{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2+4-4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

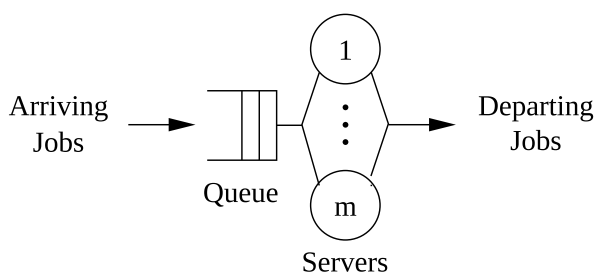
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \operatorname{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}].$$

- Utilization ρ:
  - Eine Bedieneinheit (single server) $m = 1$:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server $m > 1$:

    $m\mu$ = service rate of m servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:
  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If $\rho < 1$: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time T:
  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:
  - ➤ Time a job spends in the queue
  - ➤ response time = waiting time + service time
  - ➤ Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - ➤ number of jobs in the queue

- Number of jobs in the system K:
  - ➤ Total number of jobs in the system (queue + servers)
  - ➤ Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - ➤ Fundamental theorem of queueing theory
  - ➤ Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \,,$$
$$\overline{Q} = \lambda \overline{W} \,.$$

  - ➤ Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu},$$

Little's Law:

$$\overline{K} = \lambda\overline{T},$$

$$\overline{Q} = \lambda\overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

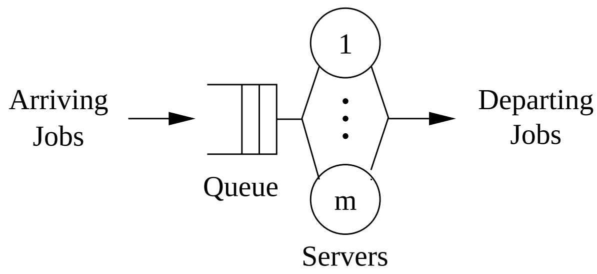
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ — $\overline{T}_A$ = mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ — $\overline{T}_B$ = mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

◆ Special distributions of ***A*** and ***B*** :

- ➤ *M* exponential distribution
- ➤ E_k Erlang-*k*-distribution
- ➤ H_k hyperexponential distribution
- ➤ *D* deterministic distribution (T_A , T_B constant)
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◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
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- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers $\boldsymbol{m = 1}$.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

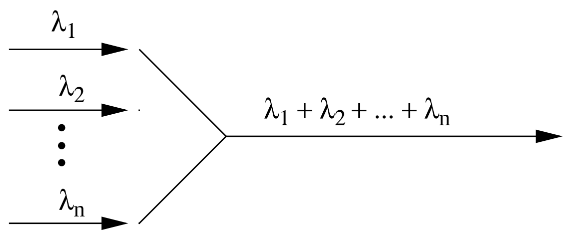
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \dfrac{1}{\lambda}\,,$
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

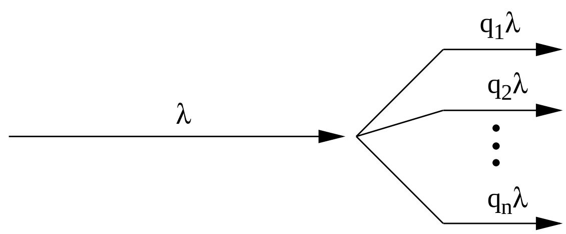
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

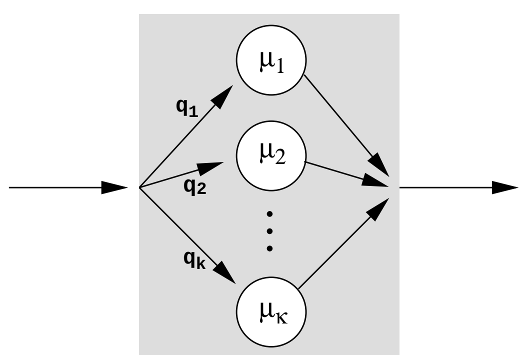


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

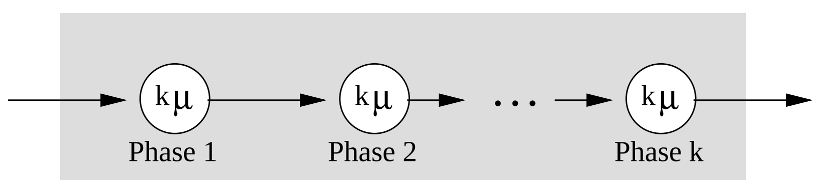
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\text{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \quad \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots.$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!} e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots ,$$
$$\overline{X} = \frac{1}{\mu} ,$$
$$\operatorname{var}(X) = \frac{1}{k\mu^2} ,$$
$$c_X = \frac{1}{\sqrt{k}} \le 1 .$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}, \qquad x \ge 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1 \mu_2}{\mu_1 - \mu_2} (e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0 ,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2} ,$$
$$\operatorname{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} ,$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1 \ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

◆ Parameters:

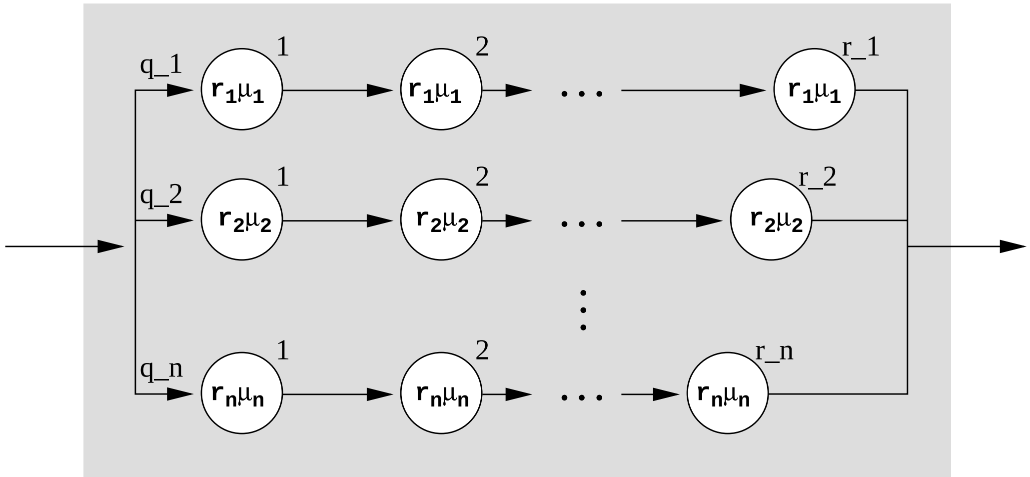
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

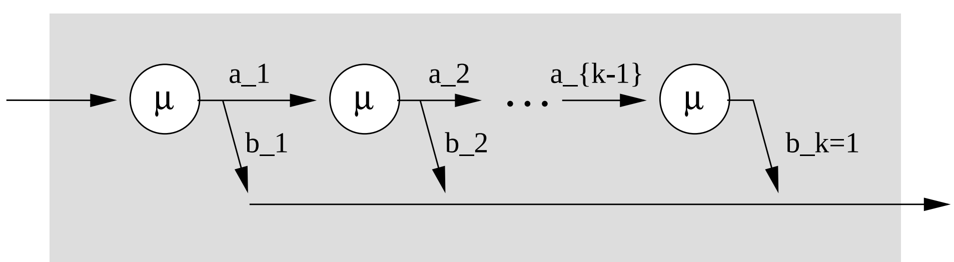
Generalized Erlang Distribution:



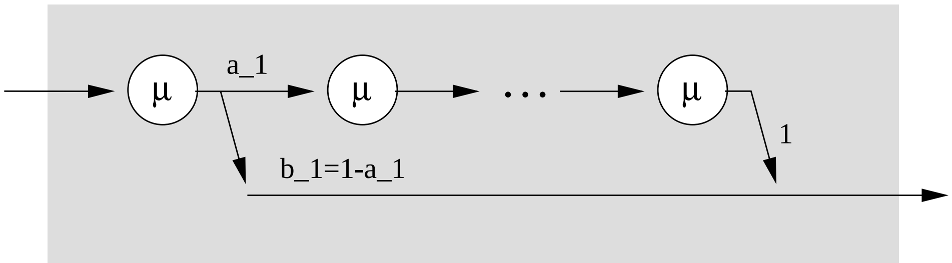
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

- Model:



$C_X < 1$:



◆ Parameters:

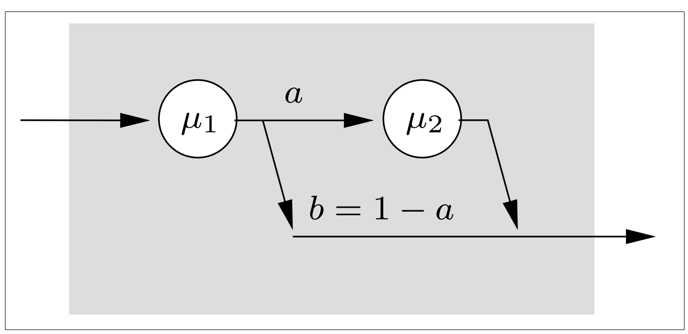
$$\overline{X} = \frac{b_1 + k(1-b_1)}{\mu},$$

$$\operatorname{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k)+k-2)}{\mu^2},$$

$$\text{ıt:} \quad c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k)+k-2)}{[b_1 + k(1-b_1)]^2}.$$

D.2 Distribution Functions

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\operatorname{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \leq 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2+4-4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

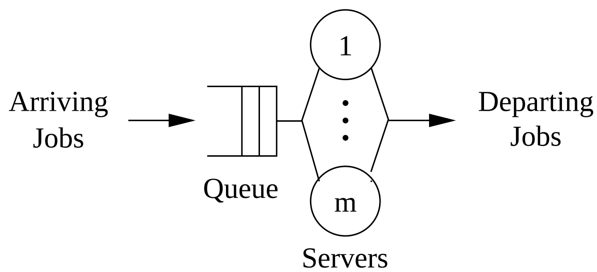
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:

  - Eine Bedieneinheit (single server) *m = 1*:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server *m > 1*:

    mμ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:

  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time *T*:

  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:
  - ➤ Time a job spends in the queue
  - ➤ response time = waiting time + service time
  - ➤ Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - ➤ number of jobs in the queue

- Number of jobs in the system K:
  - ➤ Total number of jobs in the system (queue + servers)
  - ➤ Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - ➤ Fundamental theorem of queueing theory
  - ➤ Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T}\,,$$
$$\overline{Q} = \lambda \overline{W}\,.$$

  - ➤ Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu},$$

Little's Law:

$$\overline{K} = \lambda\overline{T},$$
$$\overline{Q} = \lambda\overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

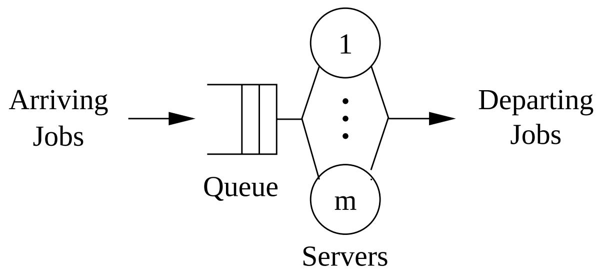
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A =$ mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B =$ mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
- ➤ *SIRO* (Service-In-Random-Order)
- ➤ *RR* (Round Robin)
- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers $m = 1$.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

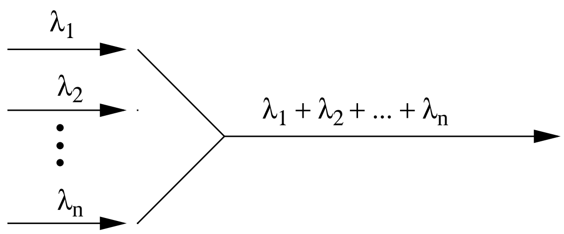
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \dfrac{1}{\lambda}\,,$
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

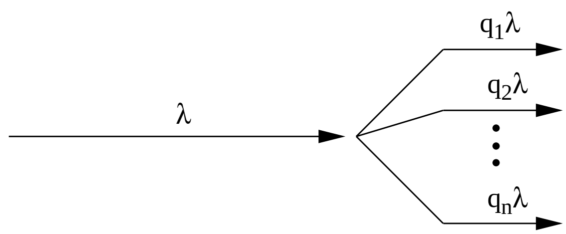
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

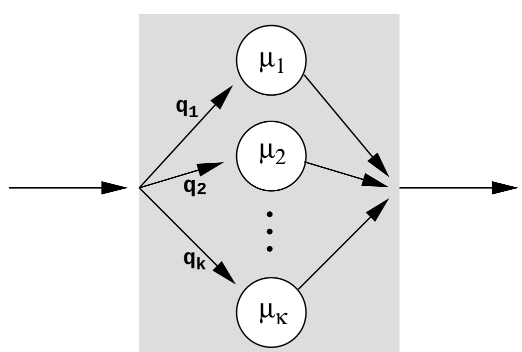


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j(1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

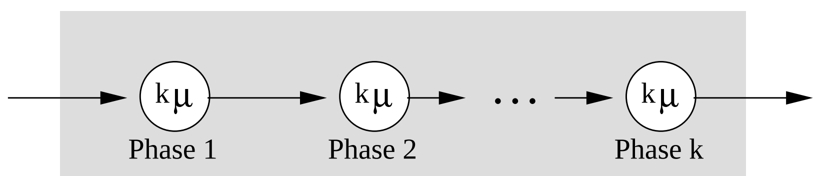
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\text{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots.$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \dots ,$$
$$\overline{X} = \frac{1}{\mu},$$
$$\text{var}(X) = \frac{1}{k\mu^2},$$
$$c_X = \frac{1}{\sqrt{k}} \le 1\,.$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \ge 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2},$$
$$\text{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2},$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

◆ Parameters:

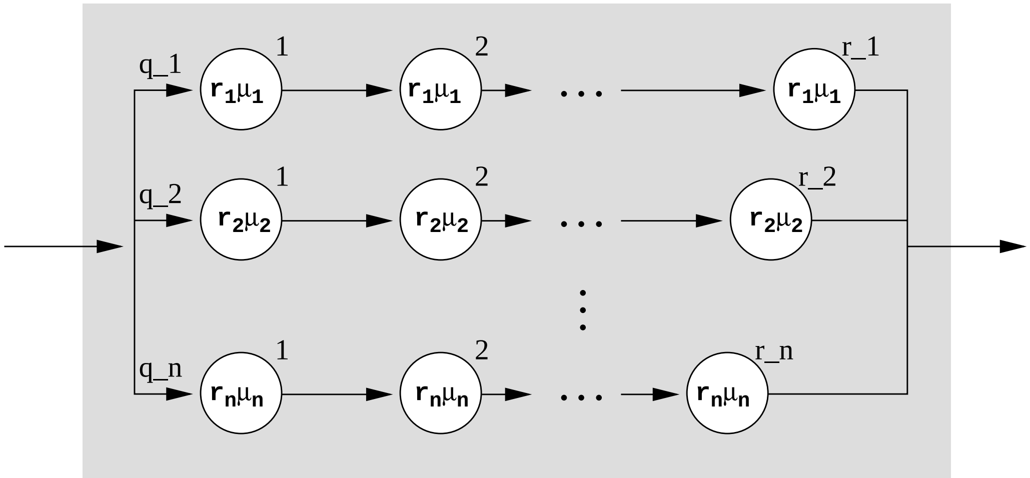
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

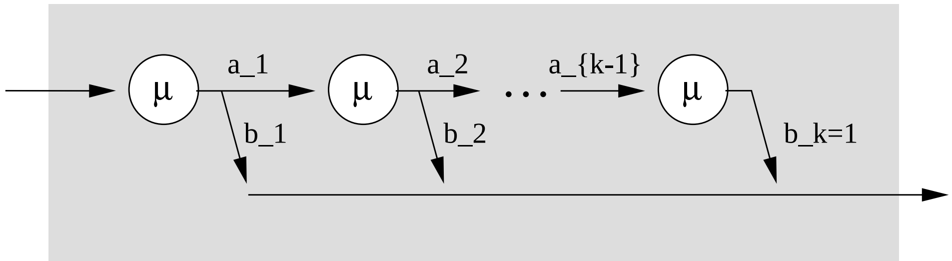
Generalized Erlang Distribution:



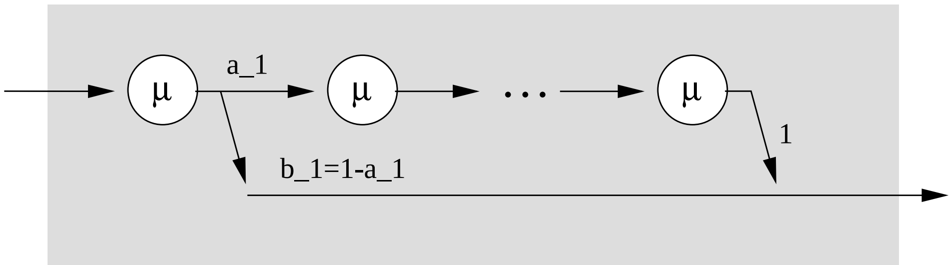
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



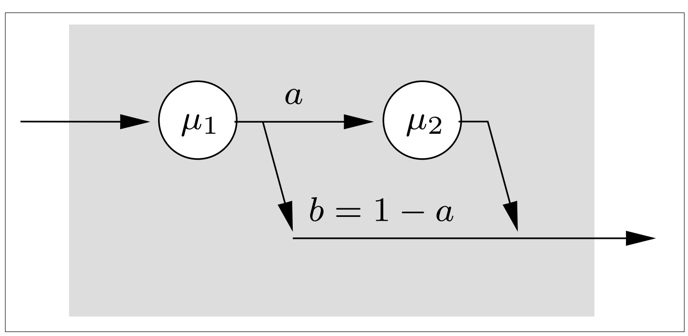
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\mathrm{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1 - b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\mathrm{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^{\alpha}), \quad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^{\alpha}), \ \lambda > 0,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right),$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

Mean:

$$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment:

$$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

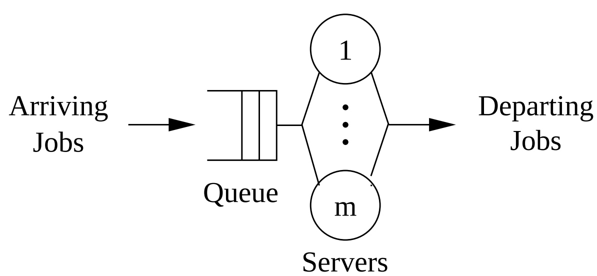
Variance:

$$\sigma_X^2 = \mathrm{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2 \,.$$

Coefficient of Variation:

$$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:
  - Eine Bedieneinheit (single server) *m* = *1*:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server *m* > *1*:

    mμ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:
  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time *T*:
  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:
  - ➤ Time a job spends in the queue
  - ➤ response time = waiting time + service time
  - ➤ Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - ➤ number of jobs in the queue

- Number of jobs in the system K:
  - ➤ Total number of jobs in the system (queue + servers)
  - ➤ Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - ➤ Fundamental theorem of queueing theory
  - ➤ Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \,,$$
$$\overline{Q} = \lambda \overline{W} \,.$$

  - ➤ Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu},$$

Little's Law:

$$\overline{K} = \lambda\overline{T},$$

$$\overline{Q} = \lambda\overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

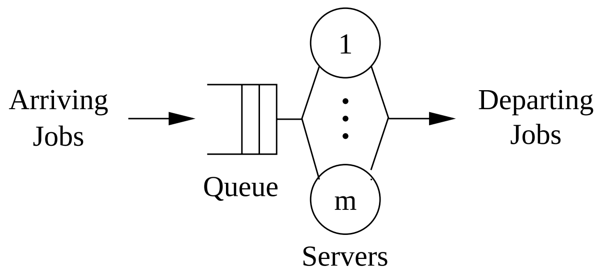
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A$ = mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B$ = mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
- ➤ *SIRO* (Service-In-Random-Order)
- ➤ *RR* (Round Robin)
- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers ***m = 1***.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\frac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \frac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \frac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

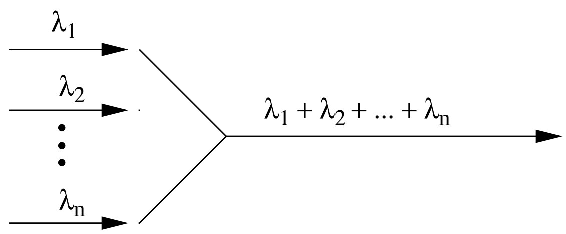
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \frac{1}{\lambda}\,,$
variance	$\text{var}(X) = \frac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

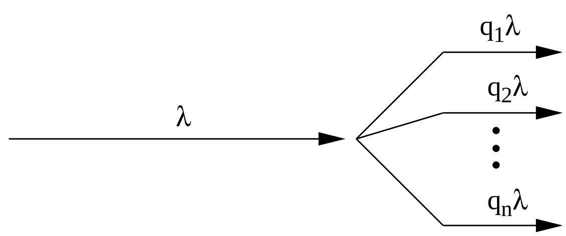
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

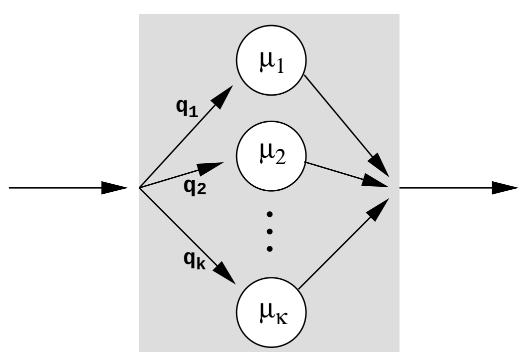


◆ Splitting:



■ **Hyperexponential Distribution**, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

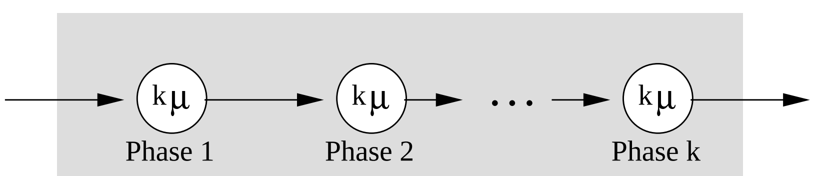
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\operatorname{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \geq 1 .$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots .$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots\ ,$$

$$\overline{X} = \frac{1}{\mu}\ ,$$

$$\mathrm{var}(X) = \frac{1}{k\mu^2}\ ,$$

$$c_X = \frac{1}{\sqrt{k}} \leq 1\ .$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\ ,$$

$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\ ,$$

$$\mathrm{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\ ,$$

$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

◆ Parameters:

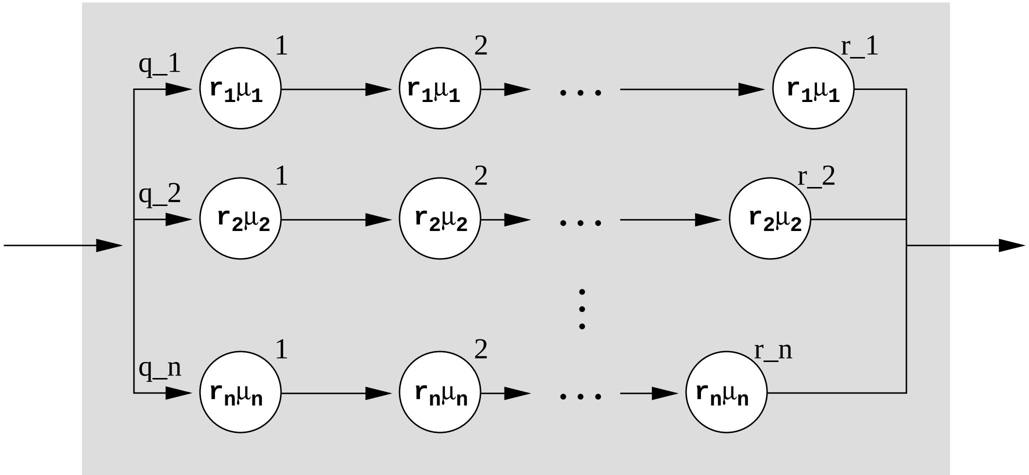
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

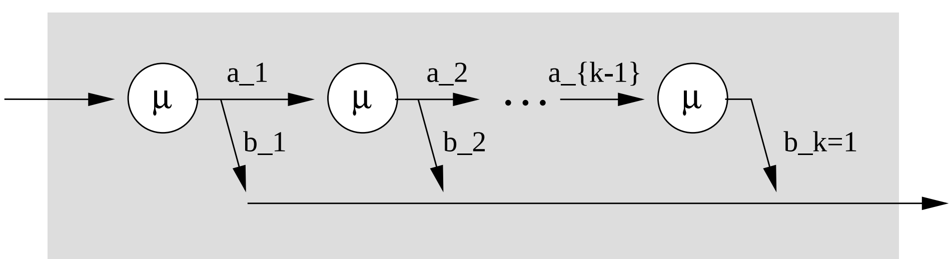
Generalized Erlang Distribution:



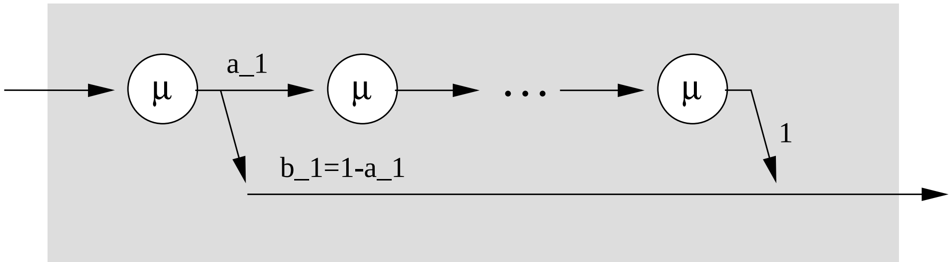
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j \mu_j (r_j \mu_j x)^{r_j - 1}}{(r_j - 1)!} \cdot e^{-r_j \mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



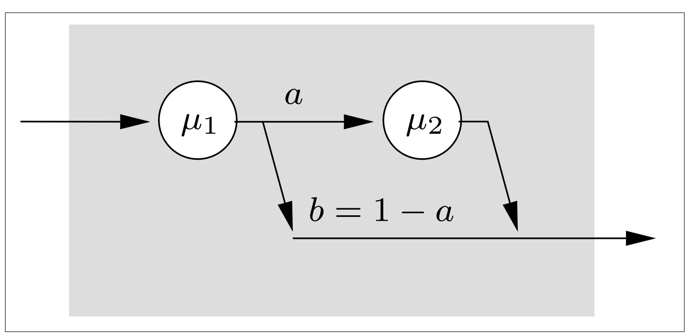
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\mathrm{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1 - b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\mathrm{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha), \quad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha), \ \lambda > 0,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right),$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\text{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $k=1,2,\ldots$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $(0<\alpha<\infty)$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $_{k=1,2,\ldots}$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $_{0<\alpha<\infty}$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\,\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\,\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\,\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\,\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\,\right]$

◆ Approximation of Mean and Variance from a sample X_i:

Mean:

$$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment:

$$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

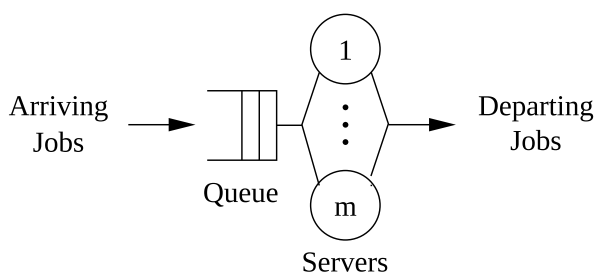
Variance:

$$\sigma_X^2 = \mathrm{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2$$

Coefficient of Variation:

$$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:

  - Eine Bedieneinheit (single server) $m = 1$:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server $m > 1$:

    $m\mu$ = service rate of m servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:

  - Number of served jobs per time unit (departure rate)
  - If $\rho < 1$: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time T:

  - Total time in system (queue + server)
  - Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \, .$$

- Little's theorem (Little's law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \, ,$$
$$\overline{Q} = \lambda \overline{W} \, .$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization: $$\rho = \frac{\lambda}{m\mu},$$

Little's Law: $$\overline{K} = \lambda\overline{T},$$ $$\overline{Q} = \lambda\overline{W}$$

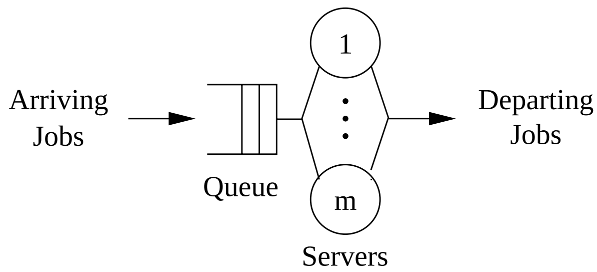
Mean Response Time: $$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs: $$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs: $$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A$ = mean interarrival time
  - service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B$ = mean service time
  - number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - *M* exponential distribution
  - E_k Erlang-*k*-distribution
  - H_k hyperexponential distribution
  - *D* deterministic distribution (T_A , T_B constant)
  - *G* general distribution
  - *GI* general independent distribution

◆ **Queueing disciplines**:

- *FCFS* (First-Come-First-Served)
- *LCFS* (Last-Come-First-Served)
- *SIRO* (Service-In-Random-Order)
- *RR* (Round Robin)
- *PS* (Processor Sharing)
- IS (Infinite Server)
- Time independent priorities
- Time dependent priorities
- Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- Interarrival time ist **exponentially** distributed
- Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- Number of servers ***m = 1***.
- Queueing dicipline: **LCFS**
- **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again
(**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

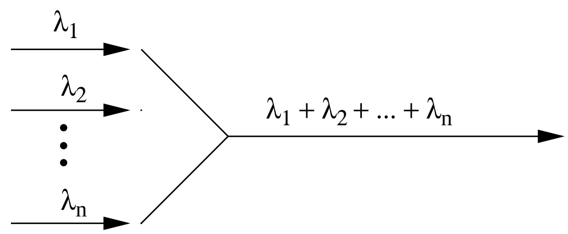
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \dfrac{1}{\lambda}\,,$
variance	$\mathrm{var}(X) = \dfrac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

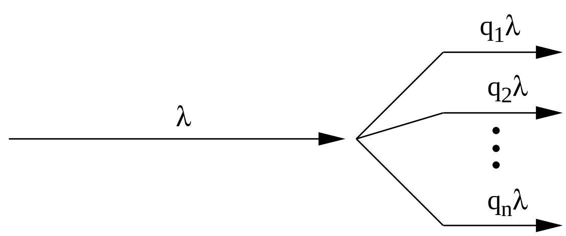
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

- Merging:

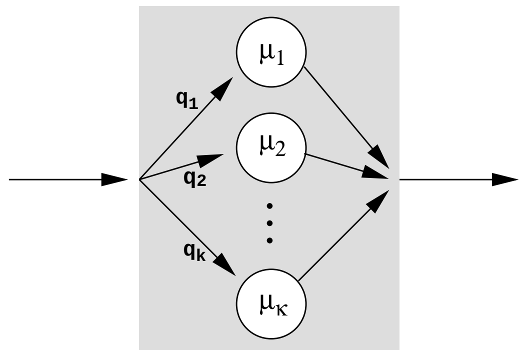


- Splitting:



Hyperexponential Distribution, H_k

- Model:



- Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

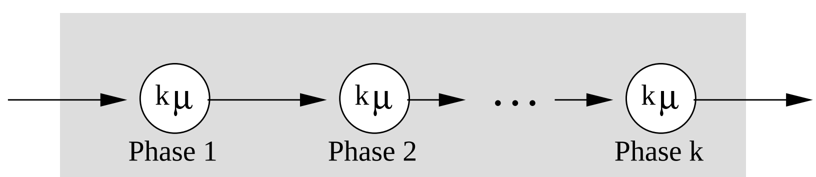
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\text{var}(X) = 2\sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots .$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!} e^{-k\mu x},\ x > 0,\ k = 1, 2, \dots,$$
$$\overline{X} = \frac{1}{\mu},$$
$$\operatorname{var}(X) = \frac{1}{k\mu^2},$$
$$c_X = \frac{1}{\sqrt{k}} \le 1 .$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}, \qquad x \ge 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1 \mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2},$$
$$\operatorname{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2},$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1 .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0.$$

◆ Parameters:

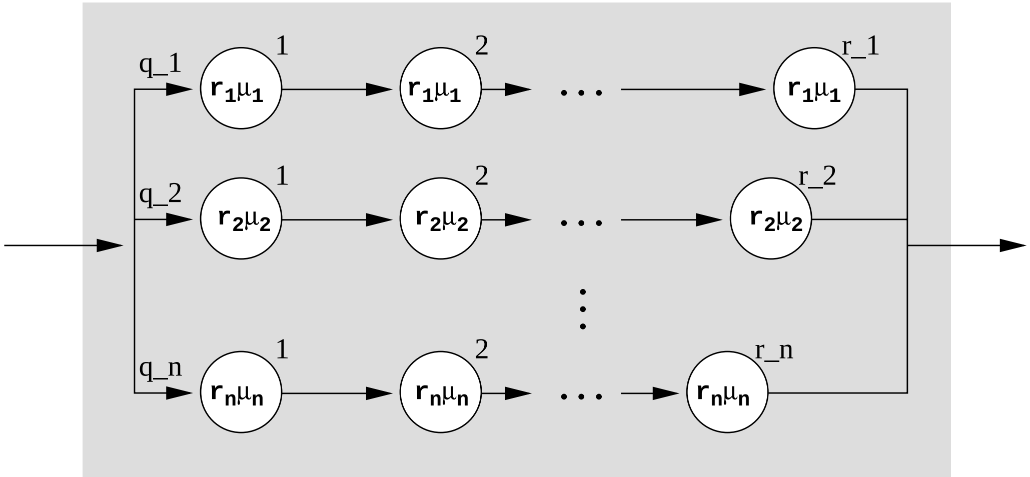
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0,$$

$$\overline{X} = \frac{1}{\mu},$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2},$$

$$c_X = \frac{1}{\sqrt{\alpha}}.$$

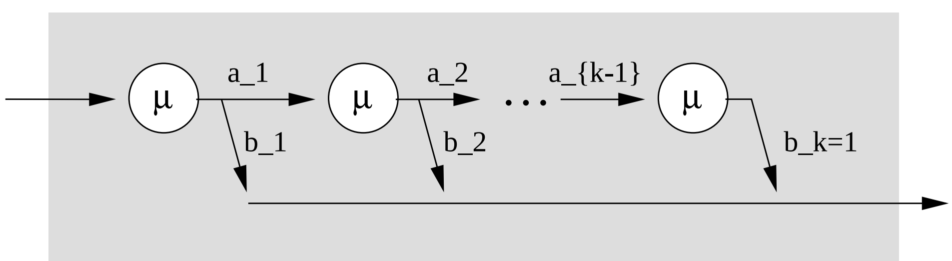
Generalized Erlang Distribution:



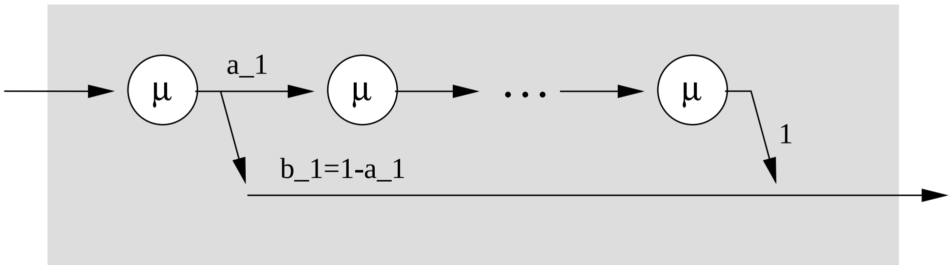
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j \mu_j (r_j \mu_j x)^{r_j - 1}}{(r_j - 1)!} \cdot e^{-r_j \mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

- Model:



$C_X < 1$:



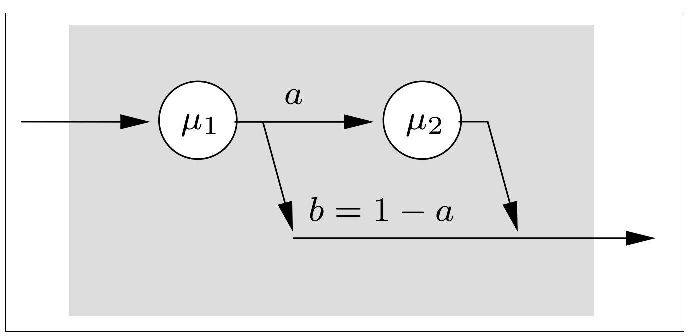
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1-b_1)}{\mu},$$

$$\text{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k)+k-2)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k)+k-2)}{[b_1 + k(1-b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\text{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha), \quad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha), \ \lambda > 0,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right),$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\text{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	$\underset{k=1,2,\ldots}{\mu, k}$	$k = \mathrm{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	$\underset{0<\alpha<\infty}{\mu, \alpha}$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1 + 2(c_X^2 - 1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2 - 1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2 - 1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \mathrm{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2 + 1)(k - 1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k - 1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1 + c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1 + c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

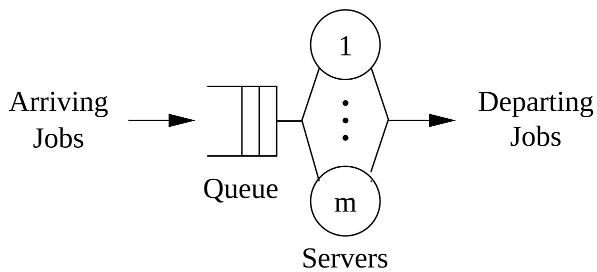
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}].$$

■ Utilization ρ:

◆ Eine Bedieneinheit (single server) $m = 1$:

$$\rho = \frac{\lambda}{\mu}.$$

◆ Multiple server $m > 1$:

$m\mu$ = service rate of m servers

$$\rho = \frac{\lambda}{m\mu},$$

◆ Condition for stability:

$$\rho < 1 \quad \longrightarrow \quad \lambda < m\mu$$

■ Throughput λ:

➤ Number of served jobs per time unit (departure rate)

➤ If $\rho < 1$: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

■ Response time T:

➤ Total time in system (queue + server)

➤ Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \, .$$

- Little’s theorem (Little’s law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \, ,$$
$$\overline{Q} = \lambda \overline{W} \, .$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu},$$

Little's Law:

$$\overline{K} = \lambda\overline{T},$$
$$\overline{Q} = \lambda\overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

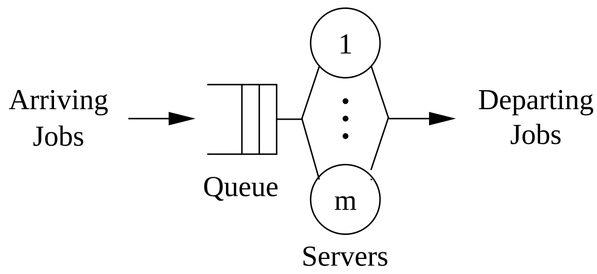
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A =$ mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B =$ mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
- ➤ *SIRO* (Service-In-Random-Order)
- ➤ *RR* (Round Robin)
- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers ***m* = *1***.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

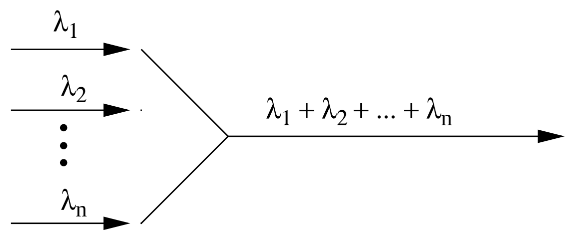
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \dfrac{1}{\lambda}\,,$
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

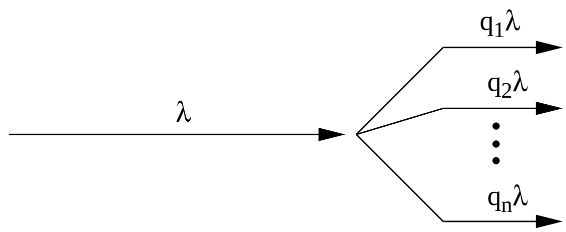
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

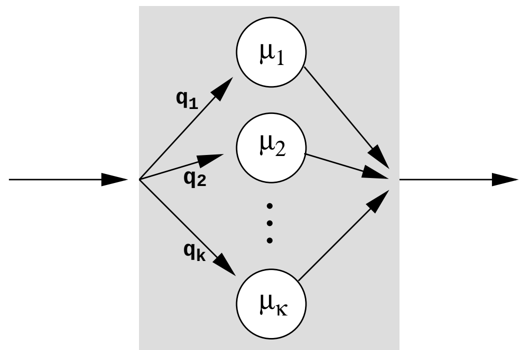


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

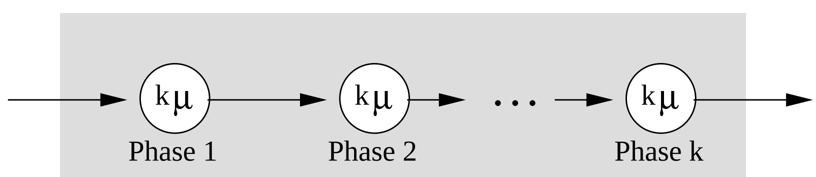
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\mathrm{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \quad \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots.$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!} e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots,$$
$$\overline{X} = \frac{1}{\mu},$$
$$\text{var}(X) = \frac{1}{k\mu^2},$$
$$c_X = \frac{1}{\sqrt{k}} \leq 1\,.$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}, \qquad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1 \mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2},$$
$$\text{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2},$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

◆ Parameters:

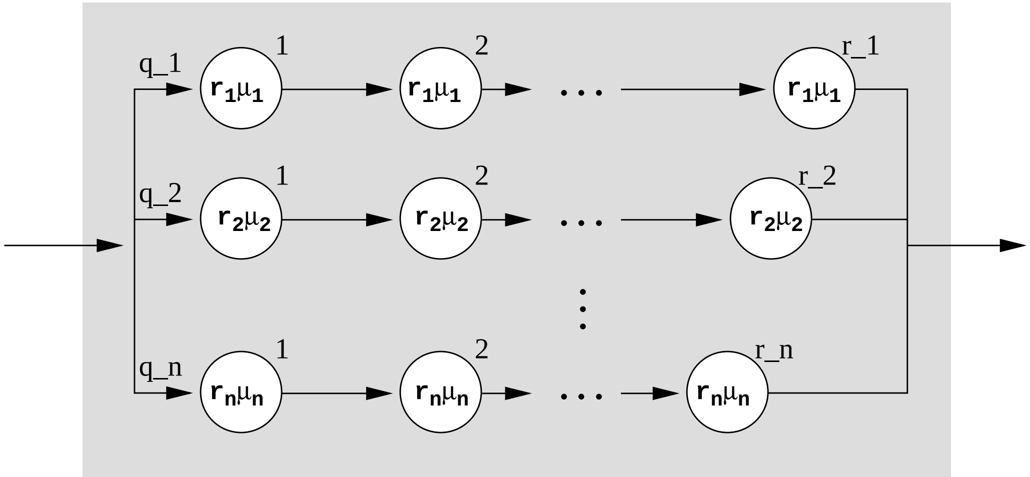
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

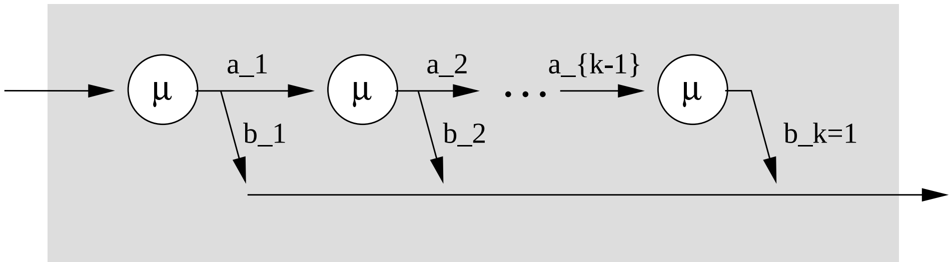
Generalized Erlang Distribution:



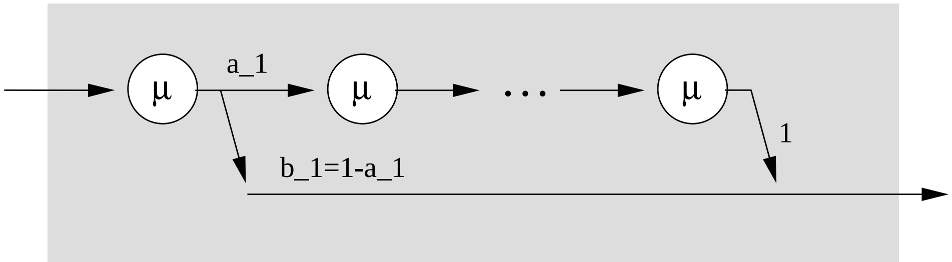
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



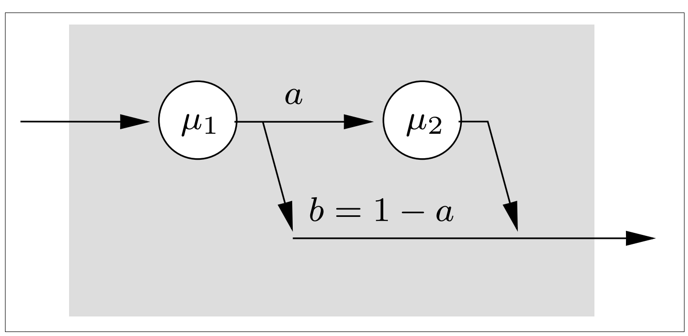
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1-b_1)}{\mu} \, ,$$

$$\text{var}\,(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2} \, ,$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1-b_1)]^2} \, .$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2} \, ,$$

$$\text{var}\,(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2} \, ,$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2} \, .$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \ \ \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1+\frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1+2/\alpha)}{\{\Gamma(1+1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2+\mu_2^2}}{\mu_1+\mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k}\frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k}\frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2\sum_{i=1}^{k}\frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

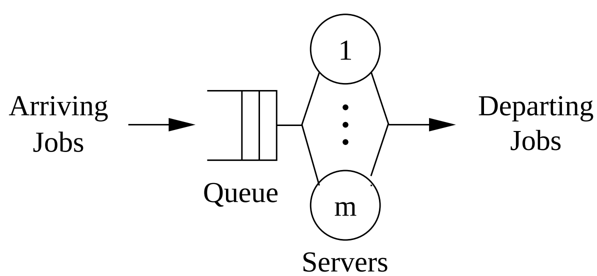
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:

  - Eine Bedieneinheit (single server) $m = 1$:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server $m > 1$:

    $m\mu$ = service rate of m servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:

  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If $\rho < 1$: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time T:

  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:

  - ➤ Time a job spends in the queue
  - ➤ response time = waiting time + service time
  - ➤ Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:

  - ➤ number of jobs in the queue

- Number of jobs in the system K:
  - ➤ Total number of jobs in the system (queue + servers)
  - ➤ Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \, .$$

- Little's theorem (Little's law):
  - ➤ Fundamental theorem of queueing theory
  - ➤ Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \, ,$$
$$\overline{Q} = \lambda \overline{W} \, .$$

  - ➤ Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization:

$$\rho = \frac{\lambda}{m\mu} \,,$$

Little's Law:

$$\overline{K} = \lambda \overline{T} \,,$$
$$\overline{Q} = \lambda \overline{W}$$

Mean Response Time:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs:

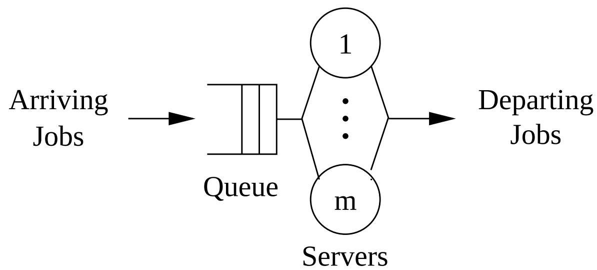
$$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A =$ mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B =$ mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time

  B distribution of the service time

  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :

  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- *FCFS* (First-Come-First-Served)
- *LCFS* (Last-Come-First-Served)
- *SIRO* (Service-In-Random-Order)
- *RR* (Round Robin)
- *PS* (Processor Sharing)
- IS (Infinite Server)
- Time independent priorities
- Time dependent priorities
- Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- Interarrival time ist **exponentially** distributed
- Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- Number of servers ***m* = *1***.
- Queueing dicipline: **LCFS**
- **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

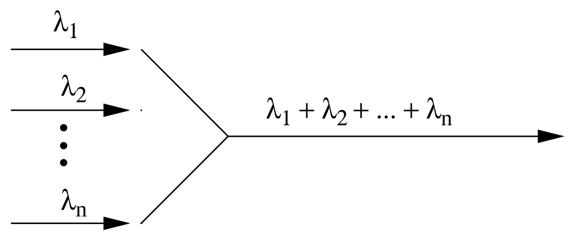
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}$,
mean	$\overline{X} = \dfrac{1}{\lambda}$,
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}$,
coefficient of variation	$c_X = 1$.

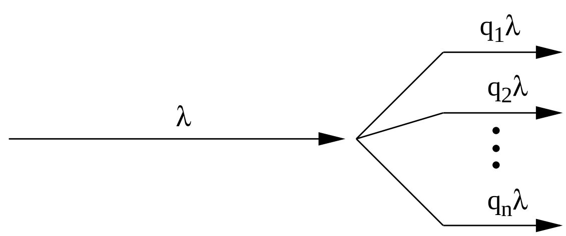
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

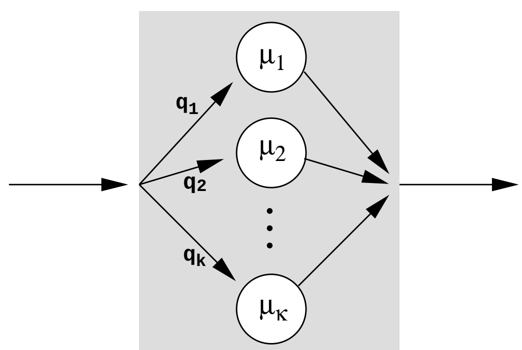


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}\,, \quad x > 0$$

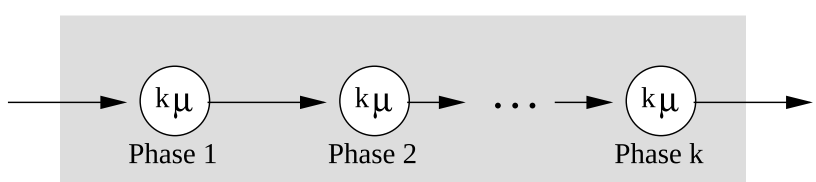
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}\,, \quad x > 0,$$

$$\mathrm{var}(X) = 2\sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2}\,,$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \quad \geq 1\,.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0,\ k = 1, 2 \ldots .$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \ldots,$$
$$\overline{X} = \frac{1}{\mu},$$
$$\mathrm{var}(X) = \frac{1}{k\mu^2},$$
$$c_X = \frac{1}{\sqrt{k}} \le 1.$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \ge 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2},$$
$$\mathrm{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2},$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1 .$$

Gamma Distribution:

- Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0, \ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

- Parameters:

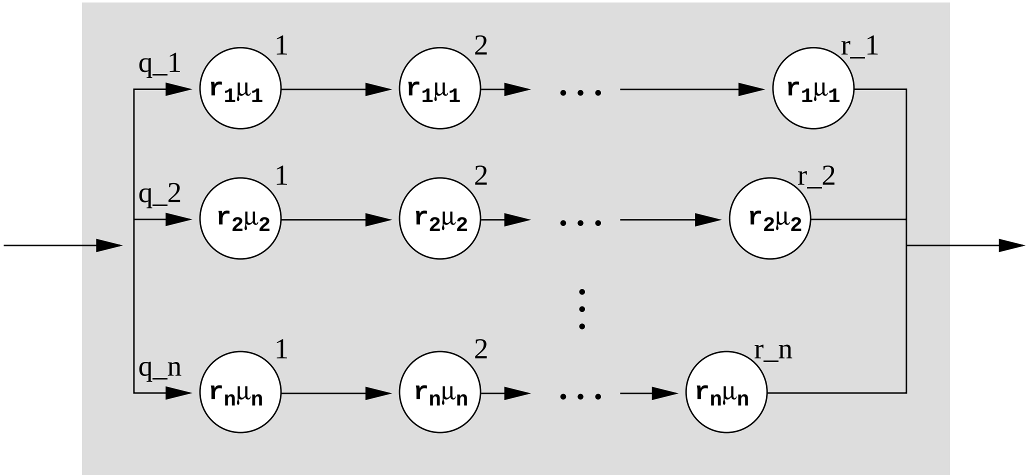
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0, \ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

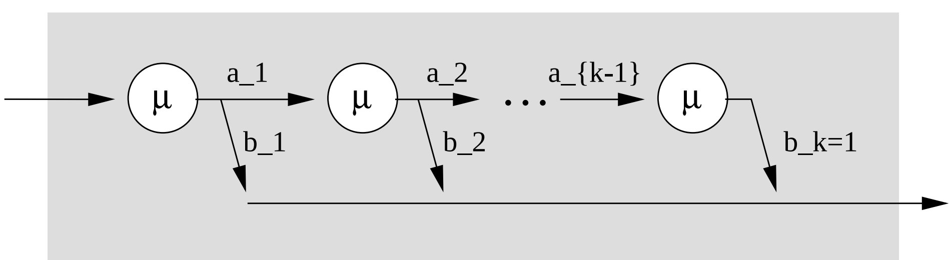
Generalized Erlang Distribution:



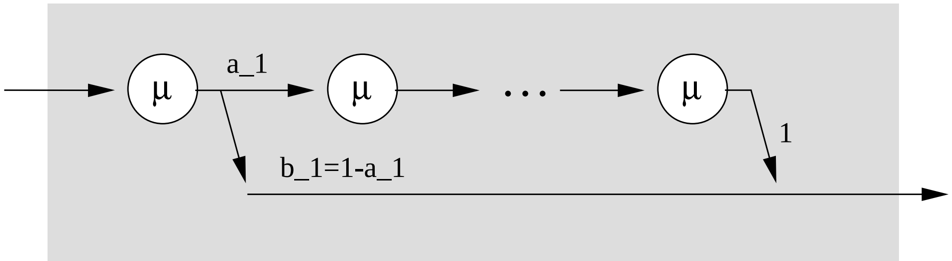
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



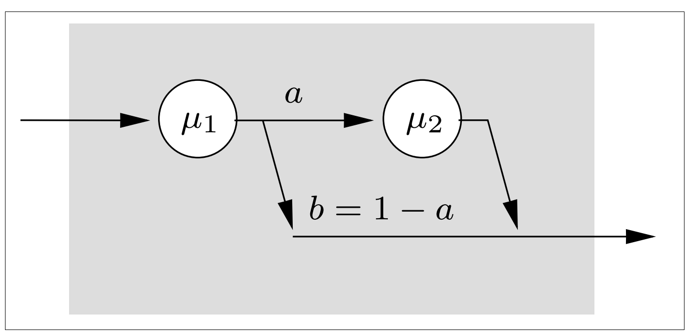
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\operatorname{var}(X) = \frac{k + b_1(k-1)\left(b_1(1-k) + k - 2\right)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\left(b_1(1-k) + k - 2\right)}{[b_1 + k(1 - b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\operatorname{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

- Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1+2/\alpha)}{\{\Gamma(1+1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $_{k=1,2,\ldots}$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $_{0<\alpha<\infty}$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

Mean:

$$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment:

$$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

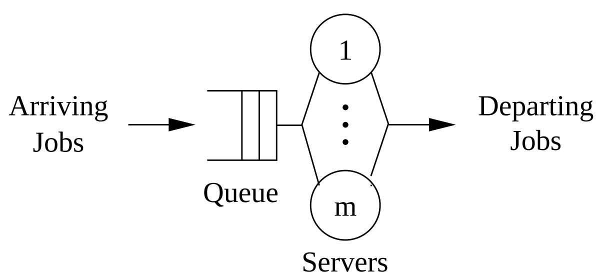
Variance:

$$\sigma_X^2 = \mathrm{var}(X) = \overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2 ,$$

Coefficient of Variation:

$$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}] .$$

- Utilization ρ:
  - Eine Bedieneinheit (single server) *m* = *1*:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server *m* > *1*:

    mμ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

D.3 Performance Measures

- Throughput λ:
  - Number of served jobs per time unit (departure rate)
  - If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time *T*:
  - Total time in system (queue + server)
  - Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \, .$$

- Little's theorem (Little's law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \, ,$$
$$\overline{Q} = \lambda \overline{W} \, .$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization: $$\rho = \frac{\lambda}{m\mu},$$

Little's Law: $$\overline{K} = \lambda\overline{T},$$ $$\overline{Q} = \lambda\overline{W}$$

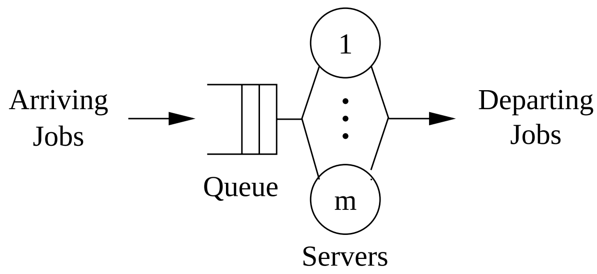
Mean Response Time: $$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs: $$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs: $$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ — $\overline{T}_A$ = mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ — $\overline{T}_B$ = mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time

  B distribution of the service time

  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
- ➤ *SIRO* (Service-In-Random-Order)
- ➤ *RR* (Round Robin)
- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers ***m = 1***.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

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$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

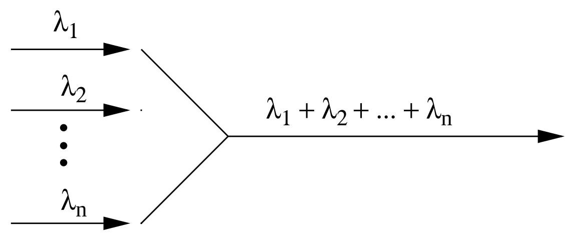
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}$,
mean	$\overline{X} = \dfrac{1}{\lambda}$,
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}$,
coefficient of variation	$c_X = 1$.

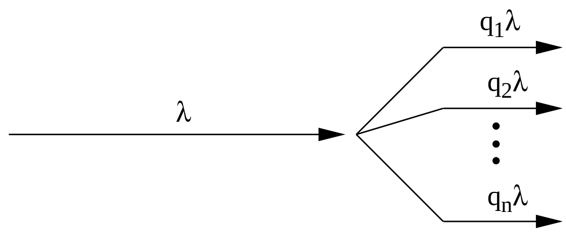
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

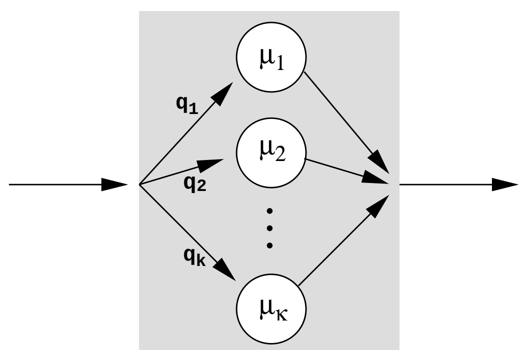


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j(1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}\,, \quad x > 0$$

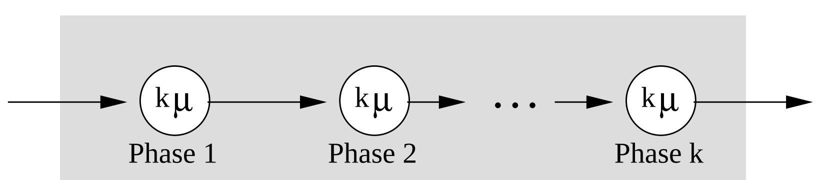
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}\,, \quad x > 0,$$

$$\mathrm{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2}\,,$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \quad \geq 1\,.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \ k = 1, 2 \ldots .$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\; x > 0,\; k = 1, 2, \ldots\,,$$
$$\overline{X} = \frac{1}{\mu}\,,$$
$$\mathrm{var}(X) = \frac{1}{k\mu^2}\,,$$
$$c_X = \frac{1}{\sqrt{k}} \le 1\,.$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \ge 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\,,$$
$$\mathrm{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\,,$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\,.$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu\cdot(\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)}\cdot e^{-\alpha\mu u}du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1}\cdot e^{-u}du, \quad \alpha > 0\,.$$

◆ Parameters:

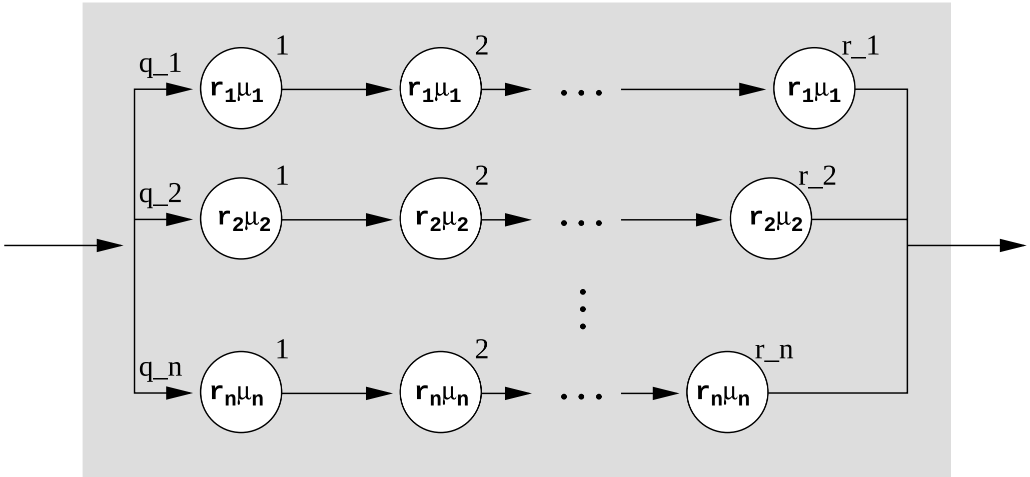
$$f_X(x) = \frac{\alpha\mu\cdot(\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)}e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

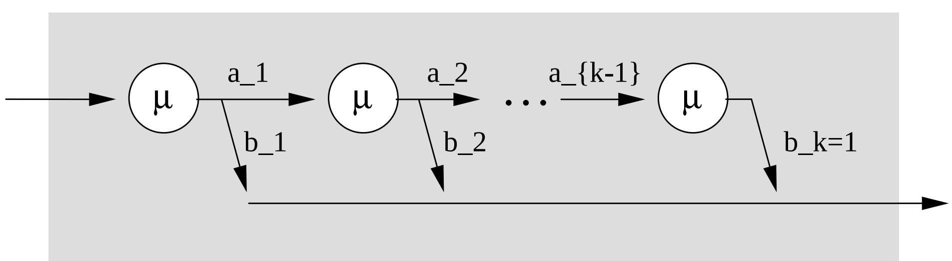
Generalized Erlang Distribution:



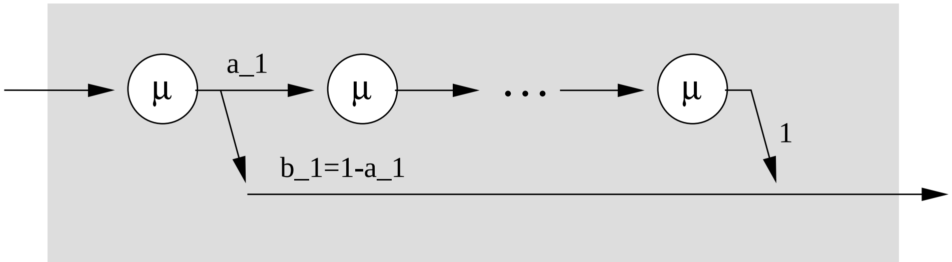
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



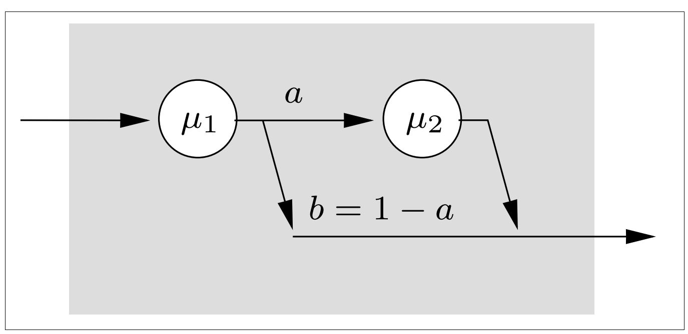
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\text{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1 - b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\text{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2 - a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2 - a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $\; _{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $\; _{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \text{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox $(c_X \le 1)$	k, b_i, μ_i	$k = \text{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2+4-4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox $(c_X > 1)$	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

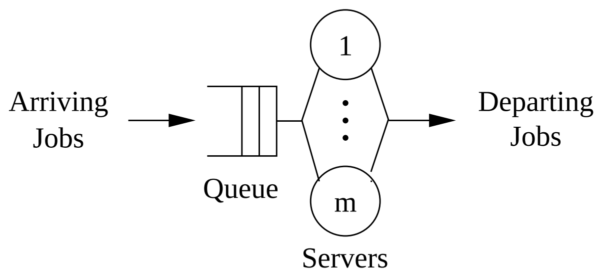
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2 \,.$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:
  - Eine Bedieneinheit (single server) *m* = *1*:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server *m* > *1*:

    $m\mu$ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:
  - Number of served jobs per time unit (departure rate)
  - If $\rho < 1$: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time *T*:
  - Total time in system (queue + server)
  - Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \,,$$
$$\overline{Q} = \lambda \overline{W} \,.$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization: $\rho = \dfrac{\lambda}{m\mu}$,

Little's Law: $\overline{K} = \lambda\overline{T}$, $\overline{Q} = \lambda\overline{W}$

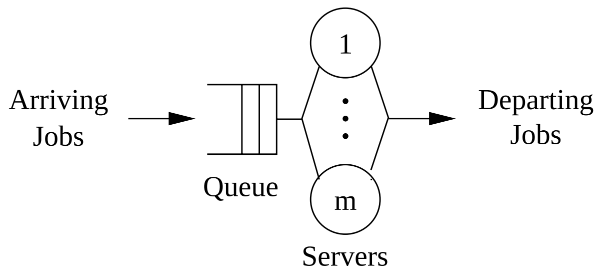
Mean Response Time: $\overline{T} = \overline{W} + \dfrac{1}{\mu}$

Mean Number of Jobs: $\overline{K} = \overline{Q} + m\rho$

Mean Number of Jobs: $\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ — $\overline{T}_A$ = mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ — $\overline{T}_B$ = mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time

  B distribution of the service time

  m number of servers

◆ Special distributions of ***A*** and ***B*** :

- ➤ *M* exponential distribution
- ➤ E_k Erlang-*k*-distribution
- ➤ H_k hyperexponential distribution
- ➤ *D* deterministic distribution (T_A , T_B constant)
- ➤ *G* general distribution
- ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- ➤ *FCFS* (First-Come-First-Served)
- ➤ *LCFS* (Last-Come-First-Served)
- ➤ *SIRO* (Service-In-Random-Order)
- ➤ *RR* (Round Robin)
- ➤ *PS* (Processor Sharing)
- ➤ IS (Infinite Server)
- ➤ Time independent priorities
- ➤ Time dependent priorities
- ➤ Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- ➤ Interarrival time ist **exponentially** distributed
- ➤ Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- ➤ Number of servers $m = 1$.
- ➤ Queueing dicipline: **LCFS**
- ➤ **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- ➤ **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

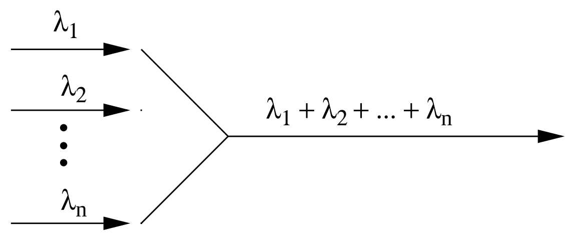
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}$,
mean	$\overline{X} = \dfrac{1}{\lambda}$,
variance	$\text{var}(X) = \dfrac{1}{\lambda^2}$,
coefficient of variation	$c_X = 1$.

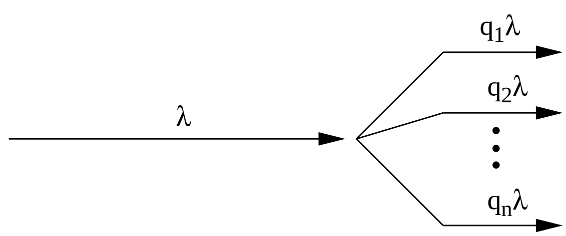
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

- Merging:

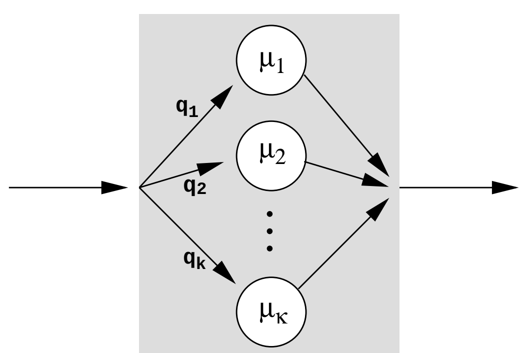


- Splitting:



Hyperexponential Distribution, H_k

- Model:



- Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j (1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

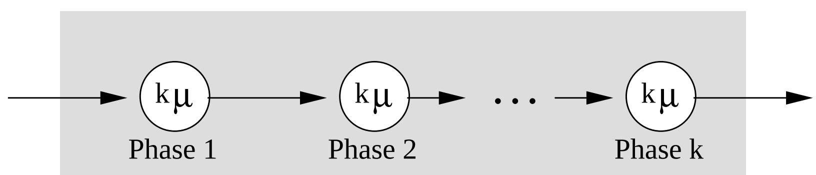
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\mathrm{var}(X) = 2\sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0,\ k = 1, 2 \ldots.$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!}e^{-k\mu x},\ x > 0,\ k = 1, 2, \dots ,$$
$$\overline{X} = \frac{1}{\mu},$$
$$\mathrm{var}(X) = \frac{1}{k\mu^2},$$
$$c_X = \frac{1}{\sqrt{k}} \leq 1\,.$$

Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1}e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1}e^{-\mu_2 x}, \qquad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1\mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2},$$
$$\mathrm{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2},$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int\limits_0^x \frac{\alpha\mu\cdot(\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)}\cdot e^{-\alpha\mu u}du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1}\cdot e^{-u}du, \quad \alpha > 0\,.$$

◆ Parameters:

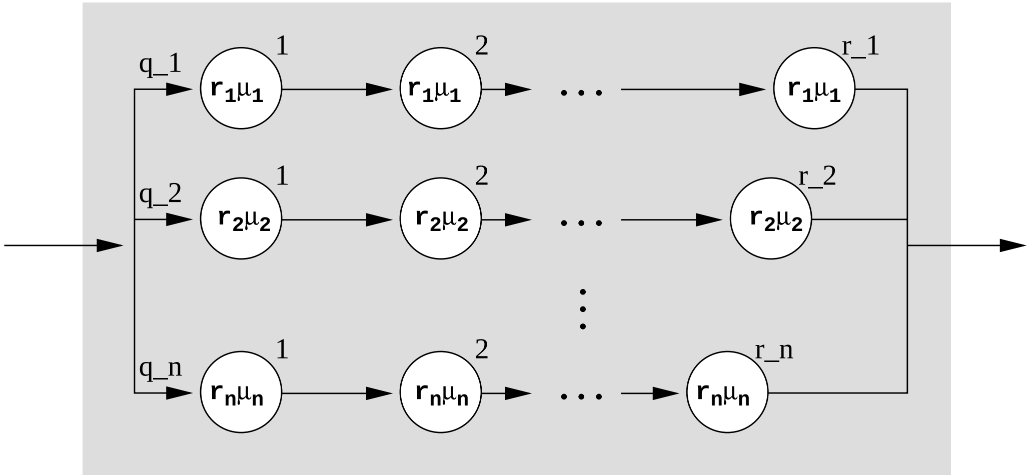
$$f_X(x) = \frac{\alpha\mu\cdot(\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)}e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

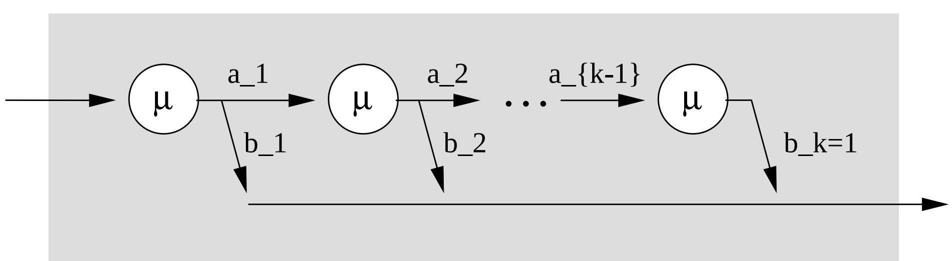
Generalized Erlang Distribution:



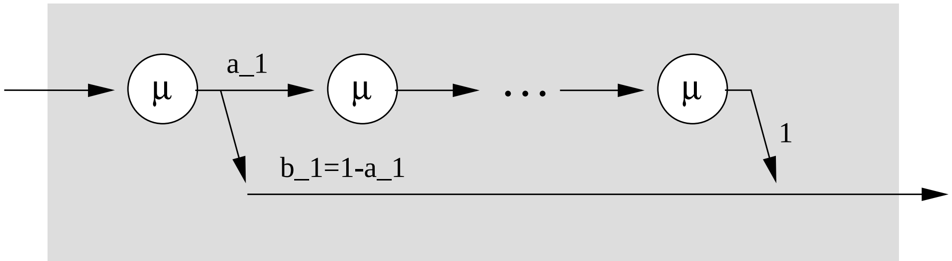
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



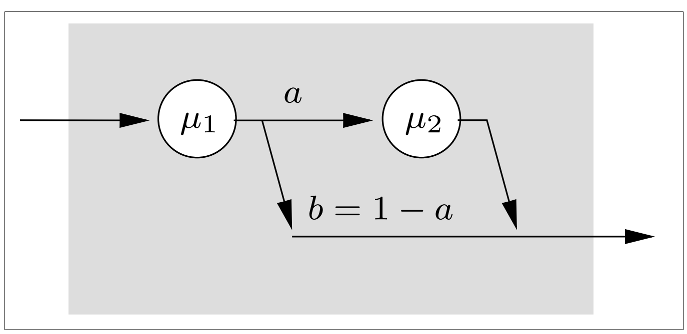
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\operatorname{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1 - b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\operatorname{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

- Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1 + 2/\alpha)}{\{\Gamma(1 + 1/\alpha)\}^2} - 1\,.$$

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $k=1,2,\ldots$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $(0<\alpha<\infty)$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k} \frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2 \sum_{i=1}^{k} \frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	$\underset{k=1,2,\ldots}{\mu, k}$	$k = \mathrm{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	$\underset{0<\alpha<\infty}{\mu, \alpha}$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1 + 2(c_X^2 - 1)}\,\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2 - 1}{2}}\,\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2 - 1}{2}}\,\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \le 1$)	k, b_i, μ_i	$k = \mathrm{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2 + 1)(k - 1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1 + c_X^2}}\,\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1 + c_X^2}}\,\right]$

◆ Approximation of Mean and Variance from a sample X_i:

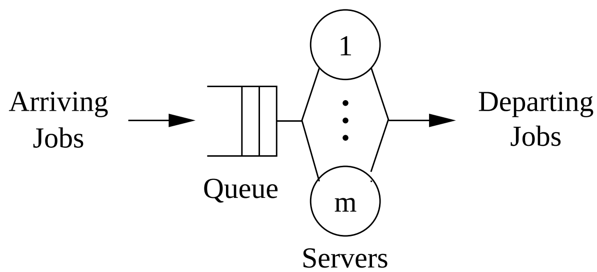
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2 ,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}] .$$

- Utilization ρ:
  - ◆ Eine Bedieneinheit (single server) *m = 1*:

$$\rho = \frac{\lambda}{\mu}.$$

  - ◆ Multiple server *m > 1*:

    mμ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

  - ◆ Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:
  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time *T*:
  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T} \,,$$
$$\overline{Q} = \lambda \overline{W} \,.$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization: $$\rho = \frac{\lambda}{m\mu},$$

Little's Law: $$\overline{K} = \lambda\overline{T},$$ $$\overline{Q} = \lambda\overline{W}$$

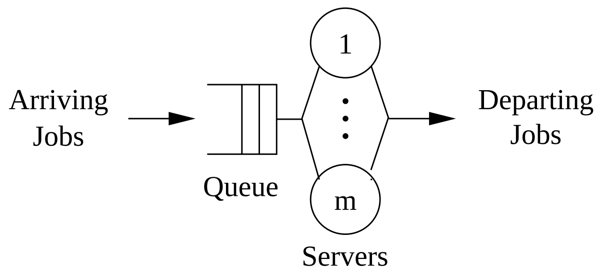
Mean Response Time: $$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs: $$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs: $$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$

D Queueing Systems

D.1 Description (Kendall's Notation)



- **Parameter**:
  - ➤ arrival rate: $\lambda = 1/\overline{T}_A$ $\overline{T}_A$ = mean interarrival time
  - ➤ service rate: $\mu = 1/\overline{T}_B$ $\overline{T}_B$ = mean service time
  - ➤ number of Servers: *m*

- **Kendall's Notation:**

  A/B/m - queueing discipline

  A distribution of the interarrival time
  B distribution of the service time
  m number of servers

  ◆ Special distributions of ***A*** and ***B*** :
  - ➤ *M* exponential distribution
  - ➤ E_k Erlang-*k*-distribution
  - ➤ H_k hyperexponential distribution
  - ➤ *D* deterministic distribution (T_A , T_B constant)
  - ➤ *G* general distribution
  - ➤ *GI* general independent distribution

◆ **Queueing disciplines**:

- *FCFS* (First-Come-First-Served)
- *LCFS* (Last-Come-First-Served)
- *SIRO* (Service-In-Random-Order)
- *RR* (Round Robin)
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- Time dependent priorities
- Preemption

◆ **Example**:

M/G/1 - LCFS Preemptive Resume (PR)

- Interarrival time ist **exponentially** distributed
- Service time is **arbitrarily** distributed (given by the ditribution function or by the mean value and the varince)
- Number of servers ***m* = *1***.
- Queueing dicipline: **LCFS**
- **Preemptive** means, that an arriving job preempts the job which is currently served in the server
- **Resume** means, that the preempted job will continued from the point where it was preempted when he gets the server again (**Repeat** means, that the job starts again at the beginning when it is preempted)

D.2 Distribution Functions

Exponential Distribution

◆ Distribution Function:

$$F_X(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\overline{X}}\right), & 0 \le x < \infty\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

$$\text{with } \overline{X} = \begin{cases} \dfrac{1}{\lambda}, & \text{if } X \text{ represents interarrival times}\,, \\ \dfrac{1}{\mu}, & \text{if } X \text{ represents service times.} \end{cases}$$

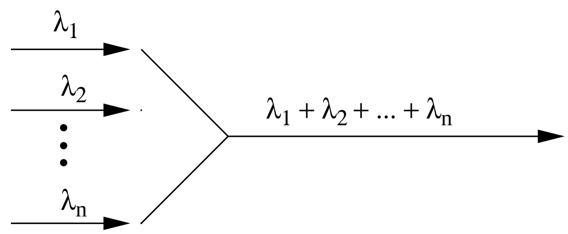
◆ Parameters:

pdf	$f_X(x) = \lambda e^{-\lambda x}\,,$
mean	$\overline{X} = \dfrac{1}{\lambda}\,,$
variance	$\mathrm{var}(X) = \dfrac{1}{\lambda^2}\,,$
coefficient of variation	$c_X = 1\,.$

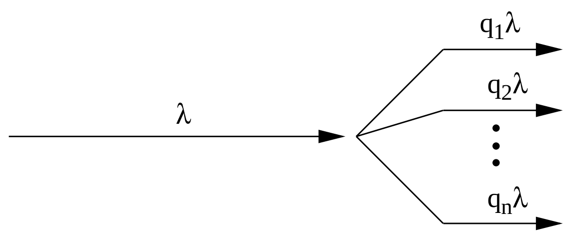
◆ Memoryless property(Markov property)

$$P(X \le u + t \mid X > u) = 1 - \exp\left(-\frac{t}{\overline{X}}\right) = P(X \le t)$$

◆ Merging:

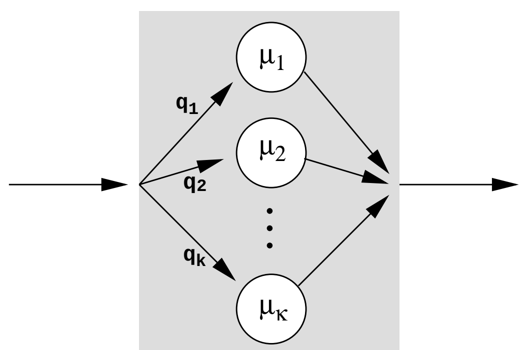


◆ Splitting:



Hyperexponential Distribution, H_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = \sum_{j=1}^{k} q_j(1 - e^{-\mu_j x}), \qquad x \geq 0$$

◆ Parameters:

$$f_X(x) = \sum_{j=1}^{k} q_j \mu_j e^{-\mu_j x}, \quad x > 0$$

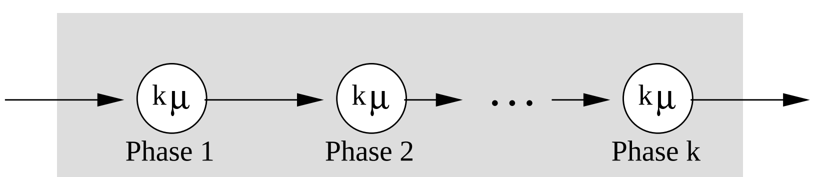
$$\overline{X} = \sum_{j=1}^{k} \frac{q_j}{\mu_j} = \frac{1}{\mu}, \quad x > 0,$$

$$\text{var}(X) = 2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - \frac{1}{\mu^2},$$

$$c_X = \sqrt{2\mu^2 \sum_{j=1}^{k} \frac{q_j}{\mu_j^2} - 1} \quad \geq 1.$$

Erlang-k-Distribution, E_k

◆ Model:



◆ Distribution Function:

$$F_X(x) = 1 - e^{-k\mu x} \cdot \sum_{j=0}^{k-1} \frac{(k\mu x)^j}{j!}, \quad x \geq 0, \; k = 1, 2 \ldots .$$

◆ Parameters:

$$f_X(x) = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!} e^{-k\mu x},\; x > 0,\; k = 1, 2, \ldots\,,$$
$$\overline{X} = \frac{1}{\mu}\,,$$
$$\text{var}(X) = \frac{1}{k\mu^2}\,,$$
$$c_X = \frac{1}{\sqrt{k}} \leq 1\,.$$

■ Hypoexponential Distribution:

◆ Distribution Function:

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}, \qquad x \geq 0.$$

◆ Parameters:

$$f_X(x) = \frac{\mu_1 \mu_2}{\mu_1 - \mu_2}(e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x > 0\,,$$
$$\overline{X} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\,,$$
$$\text{var}(X) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\,,$$
$$c_X = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1\ .$$

Gamma Distribution:

◆ Distribution Function:

$$F_X(x) = \int_0^x \frac{\alpha\mu \cdot (\alpha\mu u)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\mu u} du, \quad x \geq 0,\ \alpha > 0\,,$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du, \quad \alpha > 0\,.$$

D.2 Distribution Functions

◆ Parameters:

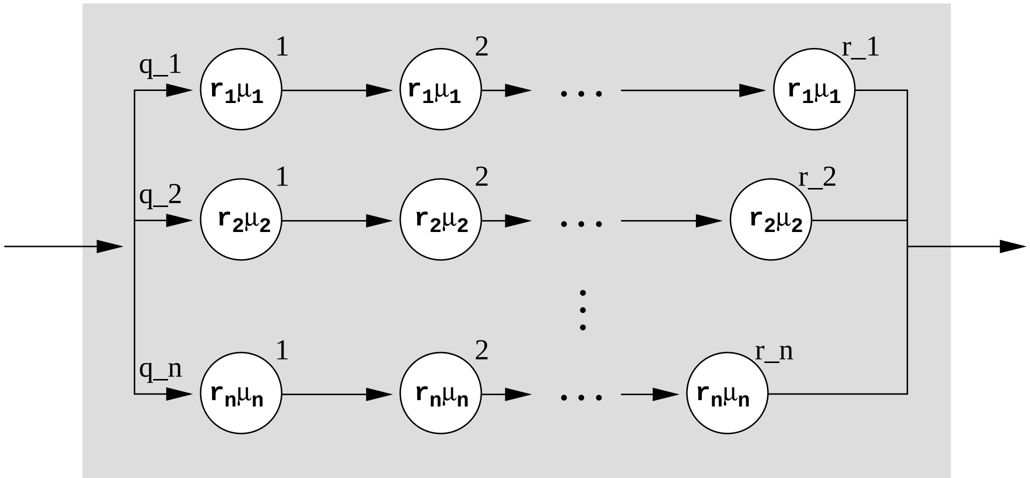
$$f_X(x) = \frac{\alpha\mu \cdot (\alpha\mu x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\mu x}, \quad x > 0,\ \alpha > 0\,,$$

$$\overline{X} = \frac{1}{\mu}\,,$$

$$\text{var}(X) = \frac{1}{\alpha\mu^2}\,,$$

$$c_X = \frac{1}{\sqrt{\alpha}}\,.$$

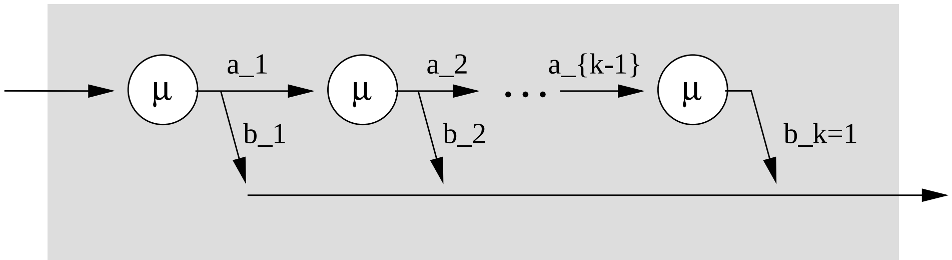
Generalized Erlang Distribution:



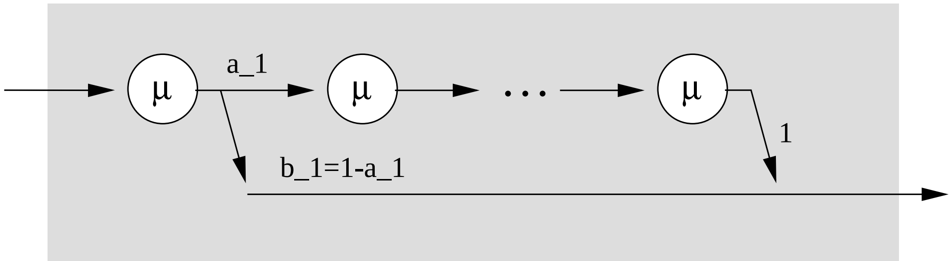
$$f_X(x) = \sum_{j=1}^{n} q_j \cdot \frac{r_j\mu_j(r_j\mu_j x)^{r_j-1}}{(r_j-1)!} \cdot e^{-r_j\mu_j x}, \qquad x \geq 0.$$

Cox Distribution, C_k (Branching Erlang Distribution)

◆ Model:



$C_X < 1$:



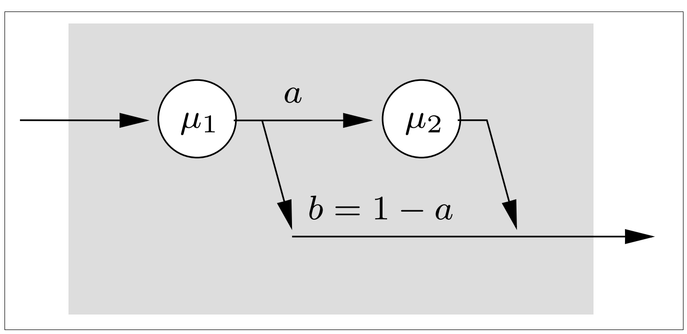
◆ Parameters:

$$\overline{X} = \frac{b_1 + k(1 - b_1)}{\mu},$$

$$\operatorname{var}(X) = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{\mu^2},$$

ıt: $$c_X^2 = \frac{k + b_1(k-1)\,(b_1(1-k) + k - 2)}{[b_1 + k(1 - b_1)]^2}.$$

$C_X > 1$:



◆ Parameters:

$$\overline{X} = \frac{1}{\mu_1} + \frac{a}{\mu_2},$$

$$\operatorname{var}(X) = \frac{\mu_2^2 + a\mu_1^2(2-a)}{\mu_1^2 \cdot \mu_2^2},$$

$$c_X^2 = \frac{\mu_2^2 + a\mu_1^2(2-a)}{(\mu_2 + a\mu_1)^2}.$$

Weibull Distribution:

$$F_X(x) = 1 - \exp(-(\lambda x)^\alpha)\,, \quad x \geq 0\,.$$

◆ Parameters:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp(-(\lambda x)^\alpha)\,, \;\; \lambda > 0\,,$$

$$\overline{X} = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right)\,,$$

$$c_X^2 = \frac{\Gamma(1+2/\alpha)}{\{\Gamma(1+1/\alpha)\}^2} - 1\,.$$

D.2 Distribution Functions

Mean, Variance and Coefficient of Variation of important Distributions:

Distribution	Parameter	$E[X]$	$\mathrm{var}(X)$	c_X
Exponential	μ	$\frac{1}{\mu}$	$\frac{1}{\mu^2}$	1
Erlang	μ, k $_{k=1,2,\ldots}$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$	$\frac{1}{\sqrt{k}} \leq 1$
Gamma	μ, α $_{(0<\alpha<\infty)}$	$\frac{\alpha}{\mu}$	$\frac{\alpha}{\mu^2}$	$0 < \frac{1}{\sqrt{\alpha}} < \infty$
Hypoexponential	μ_1, μ_2	$\frac{1}{\mu_1} + \frac{1}{\mu_2}$	$\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}$	$\frac{\sqrt{\mu_1^2+\mu_2^2}}{\mu_1+\mu_2} < 1$
Hyperexponential	k, μ_i, q_i	$\sum_{i=1}^{k}\frac{q_i}{\mu_i} = \frac{1}{\mu}$	$2\sum_{i=1}^{k}\frac{q_i}{\mu_i^2} - \frac{1}{\mu^2}$	$\sqrt{2\mu^2\sum_{i=1}^{k}\frac{q_i}{\mu_i^2} - 1} > 1$

Formulas for the Parameters of important Distributions

Distribution	Parameter	Calculation of the Parameters
Exponential	μ	$\mu = 1/\overline{X}$
Erlang	μ, k $k=1,2,\ldots$	$k = \mathrm{ceil}(1/c_X^2)$ $\mu = 1/(c_X^2 \cdot k\overline{X})$
Gamma	μ, α $0<\alpha<\infty$	$\alpha = 1/c_X^2$ $\mu = 1/\overline{X}$
Hypoexponential	μ_1, μ_2	$\mu_{1/2} = \frac{2}{\overline{X}}\left[1 \pm \sqrt{1+2(c_X^2-1)}\right]^{-1}$
Hyperexponential (H_2)	μ_1, μ_2, q_1, q_2	$\mu_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{q_2}{q_1}\frac{c_X^2-1}{2}}\right]^{-1}$ $\mu_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{q_1}{q_2}\frac{c_X^2-1}{2}}\right]^{-1}$ $q_1 + q_2 = 1, \mu_2 > 0$

Distribution	Parameter	Calculation of the Parameters
Cox ($c_X \leq 1$)	k, b_i, μ_i	$k = \mathrm{ceil}(1/c_X^2)$ $b_1 = \frac{2kc_X^2 + k - 2 - \sqrt{k^2 + 4 - 4kc_X^2}}{2(c_X^2+1)(k-1)}$ $b_2 = b_3 = \ldots = b_{k-1} = 0, \quad b_k = 1$ $\mu_1 = \mu_2 = \ldots = \mu_k = \frac{k - b_1 \cdot (k-1)}{\overline{X}}$
Cox ($c_X > 1$)	k, b, μ_1, μ_2	$k = 2$ $b = c_X^2\left[1 - \sqrt{1 - \frac{2}{1+c_X^2}}\right]$ $\mu_{1/2} = \frac{1}{\overline{X}}\left[1 \pm \sqrt{1 - \frac{2}{1+c_X^2}}\right]$

◆ Approximation of Mean and Variance from a sample X_i:

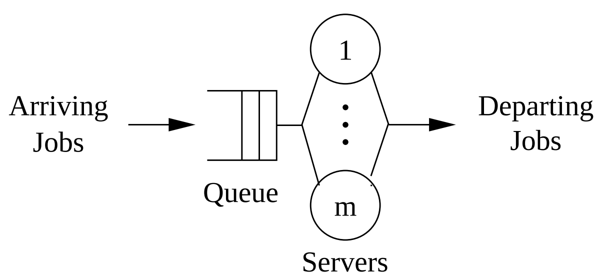
Mean: $$\overline{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

2. Moment: $$\overline{X^2} = \frac{1}{k}\sum_{i=1}^{k} X_i^2$$

Variance: $$\sigma_X^2 = \mathrm{var}(X) = \overline{(X - \overline{X})^2} = \overline{X^2} - \overline{X}^2,$$

Coefficient of Variation: $$c_X = \frac{\sigma_X}{\overline{X}}$$

D.3 Performance Measures



■ Steady state probability π_k

$$\pi_k = P[\text{there are } k \text{ jobs in the system}]\,.$$

- Utilization ρ:
  - Eine Bedieneinheit (single server) *m = 1*:

$$\rho = \frac{\lambda}{\mu}.$$

  - Multiple server *m > 1*:

    mμ = service rate of *m* servers

$$\rho = \frac{\lambda}{m\mu},$$

  - Condition for stability:

$$\rho < 1 \quad \rightarrow \quad \lambda < m\mu$$

- Throughput λ:
  - ➤ Number of served jobs per time unit (departure rate)
  - ➤ If ρ < 1: arrival rate = departure rate:

$$\lambda = m \cdot \rho \cdot \mu$$

- Response time *T*:
  - ➤ Total time in system (queue + server)
  - ➤ Sojourn time, system time

- Waiting time W:
  - Time a job spends in the queue
  - response time = waiting time + service time
  - Means:

$$\overline{T} = \overline{W} + \frac{1}{\mu}$$

- Queue length Q:
  - number of jobs in the queue

- Number of jobs in the system K:
  - Total number of jobs in the system (queue + servers)
  - Mean number of jobs in the system:

$$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k \,.$$

- Little's theorem (Little's law):
  - Fundamental theorem of queueing theory
  - Can be used to determine the mean number in the system and the mean queue length:

$$\overline{K} = \lambda \overline{T}\,,$$
$$\overline{Q} = \lambda \overline{W}\,.$$

  - Is one of the following measures known $\overline{K}$, $\overline{Q}$, $\overline{T}$ und $\overline{W}$, then the three others can be calculated.

- Important Formulas:

Utilization: $$\rho = \frac{\lambda}{m\mu},$$

Little's Law: $$\overline{K} = \lambda\overline{T},$$ $$\overline{Q} = \lambda\overline{W}$$

Mean Response Time: $$\overline{T} = \overline{W} + \frac{1}{\mu}$$

Mean Number of Jobs: $$\overline{K} = \overline{Q} + m\rho$$

Mean Number of Jobs: $$\overline{K} = \sum_{k=1}^{\infty} k \cdot \pi_k$$